Problem V.4 ... centrifuge

7 points; (chybí statistiky)

Consider a centrifuge of length $L = 30\,\mathrm{cm}$ filled with a solution in which there are homogeneously distributed small spherical particles of radius $r = 50\,\mathrm{\mu m}$ and mass $m = 5.5\cdot 10^{-10}\,\mathrm{kg}$. The density of the solution is $\rho_\mathrm{r} = 1\,050\,\mathrm{kg\cdot m^{-3}}$ and its viscosity is $\eta = 4.8\,\mathrm{mPa\cdot s}$. The container with the solution is in a horizontal position and suddenly begins to rotate at an angular velocity of $\omega = 0.5\,\mathrm{rad\cdot s^{-1}}$. Determine how long it will take for $90\,\%$ of all the particles to reach the end of the centrifuge. Do not consider interparticle collisions and movement of the particles due to diffusion. The container rotates around a vertical axis located at one of its ends.

Jarda loves to make enriched uranium.

Considering the high value of viscosity and a assumed low velocity of the particles in the centrifuge, we consider a laminar flow around the particles. Because of that, the resistance force follows the Stokes' law

$$F_\mathrm{o} = 6\pi\eta r v\,,$$

where η is the dynamic viscosity of the enviroment, r is the radius of the particles and v is their velocity.

In addition, a centrifugal force acting in the direction away from the center of the centrifuge on the particles is present. Also, we must not forget the buoyancy force, which acts in the opposite direction to the centrifugal force, analogous to submerging a body in a liquid in a homogenous gravitational field. In general, the centrifugal force has a different magnitude at every point of the particle; however, because the magnitude of the force grows linearly from the center and thanks to the symmetry of the particles, we can consider them as points of mass. Similarly, we can argue for the magnitude of the buoyancy force. Therefore, we can write their difference as

$$F_\mathrm{c} = \frac{4}{3}\pi r^3\left(\rho_\mathrm{č} - \rho_\mathrm{r}\right)\omega^2 x\,,$$

where x is their distance from the axis of the rotation, $\rho_\mathrm{č}$ is the density of the particles, ρ_r is the density of the solution, and ω is the angular velocity of the rotation of the centrifuge. We can find the density of the particles from the values given in the problem statement as $\rho_\mathrm{č} = \frac{3m}{4\pi r^3} \doteq 1050.4\,\mathrm{kg\cdot m^{-3}}$. If the density of the particles was lower than the density of the solution, the buoyancy force would overcome the centrifugal force, and the particles would follow in the direction of the center of the centrifuge.

We get a differential equation of motion

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = F_\mathrm{c} - F_\mathrm{o} = \frac{4\pi r^3\left(\rho_\mathrm{č} - \rho_\mathrm{r}\right)\omega^2}{3}x - 6\pi\eta r\frac{\mathrm{d}x}{\mathrm{d}t}\,,$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} - ax = 0\,,$$

where $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ is the acceleration, $\frac{\mathrm{d}x}{\mathrm{d}t}$ is the velocity and in order to shorten the equations we introduced constants $a = \frac{4\pi r^3(\rho_\mathrm{č}-\rho_\mathrm{r})\omega^2}{3m}$ and $b = \frac{6\pi\eta r}{m}$, where m is the mass of the particles.

Our equation is a homogenous differential equation of the second degree; therefore, we look for its solution in the form

$$x(t) = c_1\exp(\lambda_1 t) + c_2\exp(\lambda_2 t)\,,$$

where $\lambda_{1,2}$ are the solutions of the quadratic equation

$$\lambda^2 + b\lambda - a = 0 \quad \Rightarrow \quad \lambda_1 = \frac{-b + \sqrt{b^2 + 4a}}{2}, \lambda_2 = \frac{-b - \sqrt{b^2 + 4a}}{2}.$$

We can solve the problem in full generality, or we can notice that according to the values from the problem statement $b^2 \gg 4a$, therefore we can utilize the Taylor series for the square root

$$\lambda_1 = \frac{-b + b\sqrt{1 + \frac{4a}{b^2}}}{2} \approx \frac{-b + b\left(1 + \frac{2a}{b^2}\right)}{2} = \frac{a}{b}, \lambda_2 \approx -\frac{b^2 + a}{b}.$$

The exponential with the argument $\lambda_2 t$ decreases much faster than the exponential with the argument $\lambda_1 t$ grows. Hence, this term will be negligible before the particles reach the end of the centrifuge.

For now, we will keep both coefficients c_1 and c_2. Let the particle be at the distance x_0 from the center of the centrifuge and have zero velocity at time $t = 0$. The initial position of the particle x_0 gives us the condition

$$c_1 + c_2 = x_0 ,$$

while its zero velocity at the beginning of the process leads to the equation

$$\begin{aligned} \dot{x}(t) &= \lambda_1 c_1 \exp(\lambda_1 t) + \lambda_2 c_2 \exp(\lambda_2 t) , \\ \dot{x}(0) &= \lambda_1 c_1 + \lambda_2 c_2 , \\ c_1 \frac{a}{b} &= c_2 \frac{b^2 + a}{b} , \end{aligned}$$

which provides us the solution for the coefficients

$$c_1 = x_0 \frac{b^2 + a}{b^2 + 2a}, c_2 = x_0 \frac{a}{b^2 + 2a}.$$

It is evident that the coefficient c_2 is much smaller than c_1. Furthermore, the following holds $c_1 \approx x_0$. Consequently, we now see that the term with the argument λ_2 can be neglected, and for the position $x(t)$, we can write

$$x(t) = x_0 \exp \frac{a}{b} t = x_0 \exp \frac{2r^2 \left(\rho_{\check{c}} - \rho_{\mathrm{r}}\right) \omega^2}{9\eta} t .$$

This equation applies to every particle with the initial position x_0. The particles nearest to the center will reach the end latest.

We can get the looked-for time as the time in which a particle gets from the position $x_0 = L/10$ because, as we have discussed, the closer a particle was to the end of the tube at the beginning of the process, the faster it will reach its end. And since the distribution of the particles in the tube was homogenous at the beginning, 90 % of them are behind the position $L/10$ from the axis of the rotation. Thus, we need to find the time for the shift from this point. The end lies in the distance $x = L$. Therefore, the looked-for time is

$$T = \frac{9\eta}{2r^2 \left(\rho_{\check{c}} - \rho_{\mathrm{r}}\right) \omega^2} \ln 10 \doteq 2\,200\,\mathrm{d} .$$

The required time is about 2 200 days, which is more than five and a half years.

We can notice that this solution is the solution to the equation

$$b\frac{\mathrm{d}x}{\mathrm{d}t} = ax\,,$$

This corresponds to the situation where we completely neglect the acceleration term in the original equation. Because the term for the resistance force is much greater than the term a, the resistance force is always quick to compensate for the centrifugal force, and therefore, the particles only accelerate very slowly.

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