Problem V.1 ... annexation of Kaliningrad

3 points; průměr 2,62; řešilo 74 studentů

The commander of the operation to take over the Russian enclave is chilling in his recreational boat, which has the shape of a block with a base area S and height H. Suddenly, directly below him at the bottom of the Vistula Lagoon, a group of saboteurs punches a hole in the alcohol pipeline – a pipeline bringing a high-quality, scarce Czech commodity with a density ρ_B from Budějovice to Královec. Determine the conditions under which the boat sinks, assuming that it was submerged to a depth h before the accident and that the layer of beer on the surface after the accident is Δh.

Adam has a vivid imagination but doesn't want to circumvent physics with it.

The primary requirement (not explicitly stated in the problem) is that $\rho_\mathrm{B} < \rho$, where ρ_B represents the density of beer and ρ is the density of water. The problem indicates a layer of beer on the water's surface, necessitating that the density of beer is lower than that of water.

Since the boat was initially floating on the surface, its upper edge had to be above the water, ensuring that $H > h$ to prevent water from pouring inside the boat. In that case, the volume of the submerged part is Sh. According to Archimedes' principle, the buoyant force of $F_\mathrm{v} = Sh\rho g$ is pushing on the bottom of the boat in the upwards direction, where g is the gravitational acceleration. The weight of the entire vessel (including the cargo and the crew) has to be $m = Sh\rho$ because its gravitational force has balanced the buoyant force.

Once the layer of beer forms on the surface of the water, we have to split the buoyancy force into two parts – the one from the part of the boat submerged in the beer and the one from the part of the boat still in the water. Of course, if $\Delta h > H$ after the accident, the vessel is only submerged in beer. In such a case, the limiting factor is the density of the beer – if it is too low, the boat will sink. The borderline scenario occurs when the gravitational force equals the buoyant force. In that case, the depth of the submerged part is exactly H

$$mg = HS\rho_\mathrm{B} g \quad \Rightarrow \quad \frac{\rho_\mathrm{B}}{\rho} = \frac{m}{HS} = \frac{h}{H}\,.$$

If the thickness of the beer layer is greater than the height of the boat and at the same time, the density of the beer is lower than $\rho h/H$, the buoyant force is smaller than the gravitational force, and the boat sinks.

A more complicated situation occurs if $\Delta h < H$. However, it is still possible for the boat to be submerged just in beer if the depth of the submerged part is smaller than Δh. In that case, the vessel is not entirely submerged. Next, we will explore a scenario where a portion of the boat remains submerged in water.

We will split the buoyant force into two components according to the type of liquid and the volume the boat takes in the respective liquid. For the magnitude of the buoyant force, we get

$$F_\mathrm{v} = S\left(\Delta h \rho_\mathrm{B} + \rho h'\right) g\,,$$

where h' is the distance from the water-beer border to the bottom of the boat. For the boat to sink, the water has to pour inside it from the top. Therefore, the condition $\Delta h + h' = H$ must apply, and at the same time, the gravitational force of the boat must still be greater than the buoyant force. Substituting for h' into the last equation, we get

$$mg > S\left(\Delta h \rho_\mathrm{B} + \rho\left(H - \Delta h\right)\right) g \quad \Rightarrow \quad \Delta h \rho_\mathrm{B} + \rho H - \rho \Delta h < h\rho\,.$$

If we put all the densities on one side of the equation and all the vertical distances on the other side, we get a relation

$$\frac{\rho_\mathrm{B}}{\rho} < 1 - \frac{H - h}{\Delta h}\,.$$

We already know that the ratio of the densities is smaller than one. At the same time, we know that $H > h$. Therefore, even the expression on the right side has to be smaller than one. Now, it only depends on the exact magnitudes of the values. In the borderline case $\Delta h = H$, we get the same expression as if we considered $\Delta h > H$.

In the end, we can state that the results depend neither on the surface of the bottom of the boat S nor the gravitational acceleration g.

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FYKOS is organized by students of Faculty of Mathematics and Physics of Charles University. It's part of Media Communications and PR Office and is supported by Institute of Theoretical Physics of MFF UK, his employees and The Union of Czech Mathematicians and Physicists. The realization of this project was supported by Ministry of Education, Youth and Sports of the Czech Republic.