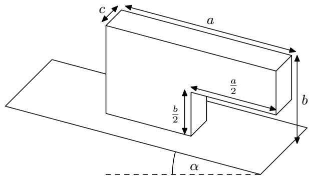
Problem III.2 ... stable sheep

3 points; průměr 2,65; řešilo 104 studentů

Consider a rectangular board and a block of wood with dimensions $a = 20\,\mathrm{cm}$, $b = 10\,\mathrm{cm}$, and $c = 5\,\mathrm{cm}$ (the shape of the inverted letter L is our approximation of a sheep). The edges of the board are parallel to the edges of the base of the block. Assuming the block tips over before sliding, at what angle will it tip over if we tilt it successively around each of the edges of the board? (See figure) *Dodo watched sheep on a hillside.*



In order to tip over, the block must reach an unstable position with its centre of gravity outside the area of the projection of its base into the vertical direction. Therefore, we must find the position of the centre of gravity.

We can think of the whole body as three smaller blocks with the centres of gravity at their geometric centres, giving us three points. We get the centre of gravity of the whole body as the average of these three points weighted by their masses

$$T[x,y] = \frac{mA[x,y] + mB[x,y] + mC[x,y]}{3m}\,,$$

where A, B, C are the points of the triangle and m is the mass of each of the blocks.

We calculate the position of the center of mass relative to the point located on the bottom edge, over which the sheep tips over. The vertical coordinate (provided the block now stands on a horizontal plane) is at the height of

$$T_y = \frac{m\frac{b}{4} + m\frac{3b}{4} + m\frac{3b}{4}}{3m} = 5.83\,\mathrm{cm}\,.$$

The horizontal position is

$$T_x = \frac{m\frac{a}{4} + m\frac{a}{4} - m\frac{a}{4}}{3m} = 1.67\,\mathrm{cm}$$

towards the back (heavier) side of the sheep.

The position of the center of gravity in the third dimension is in the middle of the shape, as expected after accounting for the symmetry. Consider that we start tilting the board with the sheep around an axis that is on the side of the "bite" (as depicted in the figure from the problem statement). We find a right triangle whose points are the center of gravity of the solid, the orthogonal projection of the center of gravity into the base of the solid, and the point O around which we will be turning the solid (it lies on the perimeter of the base). We can then find the angle α that is needed to reach the unstable position as

$$\tan\alpha_1 = \frac{T_x}{T_y}\,,$$

where T_y is the height of the center of gravity above the base and T_x is the distance of its projection into the inclined plane from the edge of the base. We get $\alpha_1 = 15.9\,^\circ$.

Similarly, if "bite" is pointing upwards, i.e. if we rotate the board around the axis on the other side of the solid, we get the condition for the angle

$$\tan\alpha_2 = \frac{\frac{a}{2} - T_x}{T_y} \quad\Rightarrow\quad \alpha_2 = 55.0\,^\circ\,.$$

We can still rotate the board around the edge parallel to the longest edge of the shape. We will thus get

$$\tan \alpha_3 = \frac{\frac{c}{2}}{T_y} \quad \Rightarrow \quad \alpha_3 = 23.2\,^\circ\,.$$

As we can see, the maximum angle at which the shape has not yet toppled over varies significantly depending on the body orientation. Sheep on a steep slope always eat with their heads uphill.

Adam Roštejnský
adam.rostejnsky@fykos.org

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