Problem II.S ... up to one's elbows

10 points; (chybí statistiky)

1. *Measure your elbow in inches. Use only your body parts for the measurement.*
2. *In ancient times, the first attempt to determine the distance of the Earth from the Sun was to measure the angular distance of the Moon from the Sun when the Moon was in the first quarter – the interface of light and darkness was direct. Determine the magnitude of this angle and compare it with the angular size of the Earth as seen from the Moon.*
3. *A laser distance meter using a He-Ne laser shows the distance exactly* $100\,\mathrm{m}$ *under standard conditions* $(20\,\mathrm{^\circ C}, 100\,\mathrm{kPa})$. *How will this value differ when the following changes:*
   - *temperature by* $30\,\mathrm{^\circ C}$
   - *pressure by* $10\,\mathrm{kPa}$
   - *a green laser with a wavelength of* $532\,\mathrm{nm}$ *will be used instead*
   - *no conversion between group and phase velocity*
4. *State at least* 4 *different ways of measuring the velocity of vehicles. Explain which physical principles are used to determine the velocity and which type of velocity it is.*

Dodo was calibrating a spectrograph.

1. The inch was historically defined as the width of the thumb; the cubit (elbow) is the distance from the elbow to the tip of the middle finger. I have executed the assigned measurement in two ways. The first method consisted of simply placing the thumb of the right hand against the forearm of the left hand. I took four measurements this way and got 20, 22, 21, and 22 inches per elbow. Since deciding where to place the thumb after removing it from the hand is not very precise – the skin on the hand is elastic, and one has to remember where to put the thumb – I decided to perform the second method by plotting the position from the elbow to the tip of the middle finger on the table. Then, while taking the measurement, I slightly pressed the thumbs of the left and right hands on the alternate hand. This procedure gave me values of 23, 23, 22, 23, and 25 inches per elbow. We can see that the second method gives higher values for no apparent reason. I calculated the average of all nine values mentioned with their standard deviation 22.3 ± 1.4. For comparison, the Czech cubit was $59.3\,\mathrm{cm}$ long, and the Old Czech inch was $24.64\,\mathrm{mm}$ long, so about 24 inches in one cubit.

2. The interface of the daylit side and the dark night side (of a planetary body) – the terminator – will be right-angled (when viewed from the Earth) if the Earth, the Moon, and the Sun make a right angle. We can calculate the angle Φ using the distances of the Earth from the Sun $A = 150 \cdot 10^6\,\mathrm{km}$ and the Moon from the Earth $a = 384 \cdot 10^3\,\mathrm{km}$ using the cosine function as

   $$\cos\Phi = \frac{a}{A}\,, \quad \rightarrow \quad \Phi = \arccos\left(\frac{a}{A}\right) \doteq 89\,^\circ\,51\,'\,.$$

   We see that it differs from the exact right angle by only about nine arc minutes, while the radius of the Earth as seen from the moon differs up to about

   $$\varphi = \arctan\left(\frac{6400\,\mathrm{km}}{384 \cdot 10^3\,\mathrm{km}}\right) \approx 1\,^\circ\,.$$

The observer's position on the Earth's surface, thus, has a much greater influence on the measured angle.

3. In this problem, we will assume that the laser is resistant to temperature and pressure changes and the air is dry. The changes in distance will arise from the changes in the refractive index of air. The value of $n-1$ in air is proportional to its density. We can express it from the ideal gas law as

$$\rho = \frac{pM_\mathrm{m}}{RT}\,,$$

where p represents the pressure, M the molar mass, R is the gas constant and T the thermodynamic temperature. From the knowledge of the refractive index n_0 at temperature T_0 and pressure p_0, we can calculate the refractive index at a different temperature and pressure as

$$n(\lambda, T, p) = 1 + (n_0(\lambda) - 1)\frac{pT_0}{p_0 T}\,,$$

where $n_0(\lambda)$ is the refractive index for the wavelength λ under reference conditions T_0, p_0. The change is due to the proportionality of $(n-1)$ to the density of air and the number of particles that interact with radiation. The helium-neon laser has a wavelength of $\lambda = 633\,\mathrm{nm}$, for which the refractive index of air is $n_0 = 1.000\,268$ at $T_0 = 20\,^\circ\mathrm{C}$ and $p_0 = 100\,\mathrm{kPa}$. We can determine the new value of length l' from the new flight of light $\Delta t'$ as

$$l' = l\frac{c\Delta t'}{n_0(\lambda)} = l\frac{c\frac{n(\lambda,T,p)}{c}}{n_0(\lambda)} = l\frac{n(\lambda,T,p)}{n_0(\lambda)}\,.$$

After substituting the dependence of the refractive index and modifying it, we get the relation for the change in the distance

$$\Delta l = l\frac{1-n_0}{n_0}\left(1 - \frac{pT_0}{p_0T}\right) \approx -l(n_0-1)\left(1 - \frac{pT_0}{p_0T}\right)\,.$$

Thus, when the temperature drops by $30\,\mathrm{K}$, we get a change in length $\Delta_T l \doteq 3.1\,\mathrm{mm}$, and when the pressure drops by $10\,\mathrm{kPa}$ the change is $\Delta_p l \doteq -2.7\,\mathrm{mm}$. If the temperature rises by a given value, then $\Delta_T l \doteq -2.5\,\mathrm{mm}$ and if the pressure increases by a given value, we get $\Delta_p l \doteq 2.7\,\mathrm{mm}$.

For the green laser, the refractive index is $n_2 = 1.000\,269\,87$ in comparison to He-Ne, where $n_1 = 1.000\,268\,243$.[1] This change is thus relatively small, especially when

$$\Delta_\lambda l = l\frac{n_1-n_2}{n_1} \approx l(n_1-n_2) = 0.16\,\mathrm{mm}\,.$$

The previous refractive index values were calculated based on phase rather than group velocities. Group refractive index can be estimated from[2] the two previous values as

$$n_g(\lambda_1) = n(\lambda_1) - \lambda_1\frac{n_1-n_2}{\lambda_1-\lambda_2} = 1.0002784\,,\quad \Delta_g l \doteq 1.0\,\mathrm{mm}\,.$$

[1] Online NIST calculator according to the Edlen equation `https://emtoolbox.nist.gov/Wavelength/Edlen.asp`

[2] Replacing the derivative in the relation in the series with a simple ratio of the differences.

Similar changes also occur, for example, when measuring the radial velocity of stars in astronomy, if the spectrograph is not maintained under stable temperature and pressure. As a consequence, such stabilization is necessary for the detection of planets with masses similar to Earth.

4. One of the most common ways to measure speed is through the average speed control. It involves measuring the time it takes for a vehicle to travel a predetermined distance s. The resulting speed, calculated as $v = s/t$, represents the vehicle's average speed over a specific road section.

   Another speed measurement option used in road transport is the traffic radar. The transmitted radio wave with a frequency of f reflects from the vehicle, and because of the Doppler effect, its frequency changes to f'. This way, we measure the instantaneous velocity of the vehicle relative to the radar position in the radial direction

   $$v = \frac{f - f'}{f}\frac{c}{2}\,.$$

   Another common way to measure speed is with a speedometer, such as in a car. In its analog form, this device determines the speed of rotation of a shaft, at the end of which is a permanent magnet. An aluminum cup surrounds this magnet on the other side, which induces eddy currents. A spring prevents the shaft from rotating too fast. As a result, the cup only rotates by an angle proportional to the speed of the shaft's rotation. The measured instantaneous rotations of the wheels are then converted into the indicated speed of the vehicle using a constant – the wheel's radius. It is worth mentioning that such measurements become meaningless if the vehicle enters into a skid.

   We can encounter more complex methods of measuring speed in aviation, where there is no contact with a solid surface. Aircraft, therefore, use Pitot tubes – devices that measure air pressure and the speed of flow using Bernoulli's principle. The speed of ships is determined similarly. We can pinpoint the speed of motion in an incompressible fluid with density ρ from the values of static pressure p_0 and dynamic pressure p_1 as:

   $$v = \sqrt{2\frac{p_0 - p_1}{\rho}}\,.$$

   In the case of compressible fluid in aircraft, the situation is more complicated by the dependence of pressure on height above the surface. In both cases, however, we can determine the relative velocity of motion with respect to the fluid. Another possibility is determining the velocity using a GPS from the position and elapsed time.

Jozef Lipták
`liptak.j@fykos.org`

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