Problem II.E ... light at the end of a tunel

12 points; průměr 9,04; řešilo 55 studentů

Measure the illumination intensity of light passing through a cola as a function of the drink's thickness. Determine the absorption coefficient by curve fitting the measured data.

A wasp flew into Jarda's soda can.

In the solution, we will first familiarize ourselves with the theory behind the experiment, provide a detailed description of the setup, present the obtained results, discuss them, and finally draw the most accurate conclusions.

Our measuring device will be the light sensor on a mobile phone placed above a container with a cola drink. The container has a transparent bottom, and beneath it, another phone with a flashlight positioned straight to the light sensor. We will gradually pour the beverage and record the changes in brightness. For comparison, we will use Kofola and Coca-Cola.

Theory

The sensor on the mobile phone records the illuminance in the unit *lux*. In the theoretical part, we will briefly introduce this unit and then outline how the illuminance could depend on the amount of the beverage in the container.

Lux is the unit of the quantity E_v, known as *illuminance*. We define this photometric quantity as the ratio of the *radiant flux* Φ_v to the area through which this flux passes. Illuminance, therefore, represents the value at each point in space. In our case, the detector area is small compared to the other dimensions of the experiment, allowing us to assume a constant illuminance on it. Thus, the radiant flux and illuminance are proportional only through the detector area.

We will also mention the photometric quantity called *luminance*. It is a property of a light source defined as the radiant flux emitted by the source within a specific solid angle. In the case of the flashlight on the mobile phone, it is evident that the highest luminance is perpendicular to the surface of the back of the mobile. Therefore, in the experiment, we will place the sensor on the second phone perpendicularly to the flashlight of the bottom mobile. Luminance is significant for us because its unit is the *candela*, one of the seven base units of the SI.

All three mentioned quantities are photometric (they relate only to the visible light spectrum). Their values are independent of the amount of light emitted or absorbed in other spectra, such as infrared or ultraviolet radiation.

However, how does the illuminance sensor reading depend on the thickness of the beverage in the container? Let us briefly describe the amount of incident light in terms of power and energy. The irradiance I, or the power incident on a unit area, decreases due to absorption in the beverage. Assuming that the loss of intensity is proportional to its magnitude through the absorption coefficient α

$$\frac{\mathrm{d}I}{\mathrm{d}x} = -\alpha I\ ,$$

we can derive the exponential dependence of the decrease of the intensity I in the material as

$$I(x) = I_0 \mathrm{e}^{-\alpha x}\,,$$

where I_0 is the intensity when the radiation hits the material (in our case the drink) and x is the distance from the edge of the material. We will further consider that the absorption is zero in the air (i.e., outside the beverage). Subsequently, we can write that the thickness of the beverage d will reduce the intensity of light passing through it to

$$I = I_0 \mathrm{e}^{-\alpha d}\,.$$

The absorption coefficient α is a function of the wavelength of light $\alpha(\lambda)$. It holds even in our case, where the white light produced by the mobile phone flashlight changes to orange after passing through the beverage. Components with bluer wavelengths are absorbed more. In our case, we will neglect this phenomenon in the derivation and focus only on how much the overall intensity of visible light decreases. The exponential relation shown is one of the possible formulations of Lambert-Beer's law, often used to analyze chemical samples based on their optical properties.

In our case, under the simplifying assumption that the cola absorbs all visible light components equally (i.e., $\alpha \neq \alpha(\lambda)$), we can consider a direct proportionality between power (intensity) and radiant flux (illuminance). Therefore, the light intensity in our model will also decrease exponentially with increasing thickness of the beverage in the container.

We expect an exponential dependence in the form of

$$E_\mathrm{v}(d) = E_{\mathrm{v}0}\mathrm{e}^{-\alpha d} + E_\mathrm{s}$$

and looking for the parameter α. We added the parameter E_s because we did not take the measurement in a completely dark room, so we observed a non-zero external illumination. The value $E_{\mathrm{v}0}$ is the direct illumination of the sensor from the other phone.

Recall that if we measure a sample of n values $x_1, x_2, x_3, \ldots, x_n$ random variable x, we can calculate its sample mean $\overline{x}$ as their sum divided by their count

$$\overline{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}\,. \tag{1}$$

Since this average is also a random variable (we measure a different n-tuple of values), it is possible to define its error as

$$s_{\overline{x}} = \sqrt{\frac{(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + (x_3 - \overline{x})^2 + \ldots + (x_n - \overline{x})^2}{n(n-1)}}\,. \tag{2}$$

Statistics tells us that the best estimate of the true value of the quantity x_exp from the measured values is then

$$x_\mathrm{exp} = (\overline{x} \pm s_{\overline{x}})\,\mathrm{j}\,,$$

where j generally denotes the unit of x.

Setting and conducting of the experiment

As mentioned above, we will experiment with a "two-phone sandwich arrangement." We have chosen the mobile phone flashlight as the light source because it provides a constant source of white light and is flat, allowing us to place a container for cola on it. The container is a transparent plastic box without a lid. We will put the second mobile phone with the light sensor above the container and directly above the light source to capture the maximum amount of signal. The upper mobile phone will be secured using its cover and rigid rods (kitchen knives, in our case) to the top edges of the container. The *phyphox* app, available for free, will be used in the *Light* measurement mode, recording light intensity in lux several times per second.

We will record the light intensity with an empty container, corresponding to the theoretical value $E_{v0} + E_s$. Subsequently, we will pour a certain amount of beverage, wait for the level to stabilize, and the phone to record sufficient data to extract the corresponding light intensity value.

To determine the amount of poured beverage as accurately as possible, we used a digital kitchen scale with a precision of 1 g, measuring 30 g of beverage each time. The total volume of the beverage in the box was approximately 400 g at the end of the experiment. We used Kofola and Coca-Cola as the beverages.

To find the height of the liquid level in the box, we needed its dimensions and the volume occupied by the 30 g of liquid. The specific dimensions are in the *Results* section. The same chapter will present the results of density measurements using a graduated cylinder. Due to its low-resolution scale, we decided not to use it directly to measure the same volume of liquid each time we poured it into the box.

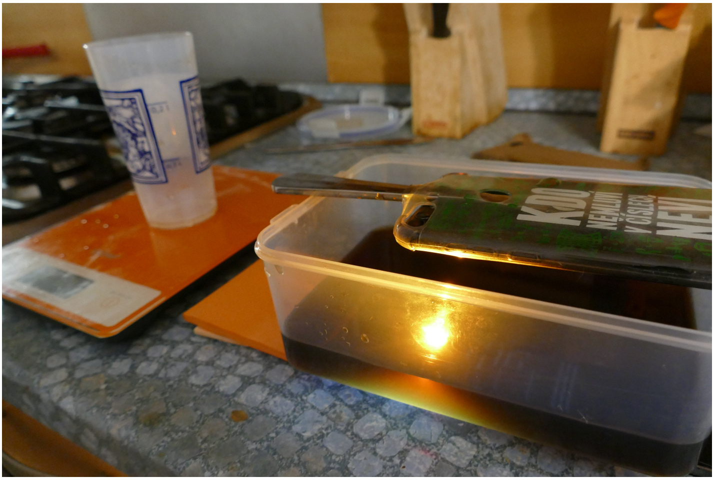

Fig. 1: Photo of experimental apparatus. On the left is the kitchen scale for reading the selected amount of drink.

Results

In this part of the solution, we will present the measured data and take the necessary steps to answer the task.

We measured each height and both liquids three times to reduce statistical uncertainty. Since the mobile phone recorded the light intensity several times per second, the values of this quantity presented below are averages over several seconds when we reached a stable state, and the recorded value changed only on the order of a few lux. We marked the order of each measurement with a number following the name of the respective liquid. The data are presented in table 1.

Tab. 1: The dependence of illuminance on the amount of drink in the container.

mass	Kofola 1	Kofola 2	Kofola 3	Coca-Cola 1	Coca-Cola 2	Coca-Cola 3
$\frac{m}{\mathrm{g}}$	$\frac{E_\mathrm{v}}{\mathrm{lux}}$	$\frac{E_\mathrm{v}}{\mathrm{lux}}$	$\frac{E_\mathrm{v}}{\mathrm{lux}}$	$\frac{E_\mathrm{v}}{\mathrm{lux}}$	$\frac{E_\mathrm{v}}{\mathrm{lux}}$	$\frac{E_\mathrm{v}}{\mathrm{lux}}$
30	9 209	9 068	9 627	9 735	9 890	1 0057
60	7 354	7 213	7 660	7 612	8 194	8 185
90	5 860	5 901	6 243	5 785	6 789	6 912
120	4 729	4 751	4 952	5 007	5 513	5 728
150	3 820	3 856	4 003	4 290	4 524	4 811
180	3 089	3 156	3 242	3 576	3 770	4 053
210	2 524	2 606	2 651	2 854	3 283	3 403
240	2 056	2 141	2 165	2 566	2 777	2 876
270	1 715	1 782	1 780	2 157	2 340	2 451
300	1 387	1 490	1 476	1 817	1 963	2 089
330	1 117	1 246	1 223	1 354	1 661	1 761
360	880	1 049	1 018	1 145	1 434	1 539
390	667	884	853	971	1 227	1 322
420	488	752	721			
450	378	647	611			

Next, we present the measured dimensions of the box in table 2. Its dimension perpendicular to the base (i.e., the height of the edges) is about 7 cm. While this quantity does not appear in any relation, it is unnecessary to measure it more precisely.

Tab. 2: The dimensions a and b of the box. Average in the penultimate line and measurement error in the last line.

measurement	$\frac{a}{\mathrm{cm}}$	$\frac{b}{\mathrm{cm}}$
1	17.9	12.9
2	18.0	13.0
3	17.9	13.0
4	18.0	13.0
5	17.9	12.9
mean	17.9	13.0
error	0.1	0.1

The total area of the box is $S = ab = (233 \pm 2)\,\mathrm{cm}^2$, where the error of the area ΔS was determined from the error propagation theory as

$$\Delta S = S\sqrt{\left(\frac{\Delta a}{a}\right)^2 + \left(\frac{\Delta b}{b}\right)^2}\,,$$

where Δa and Δb are the errors of the quantities a and b respectively.

Last but not least, the weights of the drinks as a function of volume are stated in 2. For the density measurements, we chose a kitchen measuring cup, which we gradually filled according to the traces and measured the weight of the drink.

Tab. 3: The dependence of the weight of the drink on its volume in the measuring cup.

mass	Kofola 1	Kofola 2	Kofola 3	Coca-Cola 1	Coca-Cola 2	Coca-Cola 3
$\frac{V}{\mathrm{cm}^3}$	$\frac{m}{\mathrm{g}}$	$\frac{m}{\mathrm{g}}$	$\frac{m}{\mathrm{g}}$	$\frac{m}{\mathrm{g}}$	$\frac{m}{\mathrm{g}}$	$\frac{m}{\mathrm{g}}$
0	0	0	0	0	0	0
50	45	44	48	48	44	45
75	72	71	73	70	72	72
100	98	98	96	94	95	95
125	122	120	123	123	121	118
150	150	149	151	147	145	149
175	177	175	177	173	173	174
200	202	203	203	199	200	201
225	226	226	226	225	224	224
250	253	254	252	251	252	252
275	280	281	279	276	275	278
300	308	305	306	304	303	303

We fitted the measured data for each series with a straight line in the form $m = \rho V$ where ρ is a parameter representing density. Fitting the data with such a function in the Python program yielded the densities of liquids in table 4. We determined the resulting density as the average of individual measurements. Since the standard deviation of the arithmetic mean is significantly lower than the error of measurements, we calculated the error of resulting density as the error of one measurement divided by the square root of the number of measurements.

Tab. 4: The values of the densities ρ from the fitting of the data with a straight line and their errors. The mean of the density and its error were calculated using the formulas (1) a (2).

	Kofola		Coca-Cola	
measurement	$\frac{\rho}{\mathrm{g{\cdot}cm^{-3}}}$	$\frac{\Delta\rho}{\mathrm{g{\cdot}cm^{-3}}}$	$\frac{\rho}{\mathrm{g{\cdot}cm^{-3}}}$	$\frac{\Delta\rho}{\mathrm{g{\cdot}cm^{-3}}}$
1	1.000 1	0.005	1.009	0.004
2	0.996 4	0.005	1.008	0.006
3	0.998 5	0.005	1.011	0.005
mean	0.998		1.010	
error	0.003		0.003	

The densities of both liquids are close to the density of water. Coca-Cola has a higher density, which may be related to the fact that there is more dissolved sugar in it per 100 ml.

We have everything ready to plot the dependency graphs of illuminance on surface height. We fitted these graphs (again in Python) with functions

$$E_\mathrm{v}(d) = E_{\mathrm{v}0}\mathrm{e}^{-\alpha d} + E_\mathrm{s}\,.$$

We listed the fitting parameters in the following table 5.

Tab. 5: The dependence of the illuminance on surface height.

measurement	$\frac{E_0}{\mathrm{lux}}$	$\frac{\Delta E_0}{\mathrm{lux}}$	$\frac{\alpha}{\mathrm{m}^{-1}}$	$\frac{\Delta\alpha}{\mathrm{m}^{-1}}$	$\frac{E_\mathrm{s}}{\mathrm{lux}}$	$\frac{\Delta E_\mathrm{s}}{\mathrm{lux}}$
Kofola 1	9 130	60	0.168	0.003	17	5
Kofola 2	8 730	30	0.172	0.002	290	20
Kofola 3	9 370	20	0.176	0.001	240	20
Coca-Cola 1	8 970	20	0.178	0.001	530	20
Coca-Cola 2	9 450	70	0.161	0.004	470	80
Coca-Cola 3	9 560	60	0.153	0.002	430	60

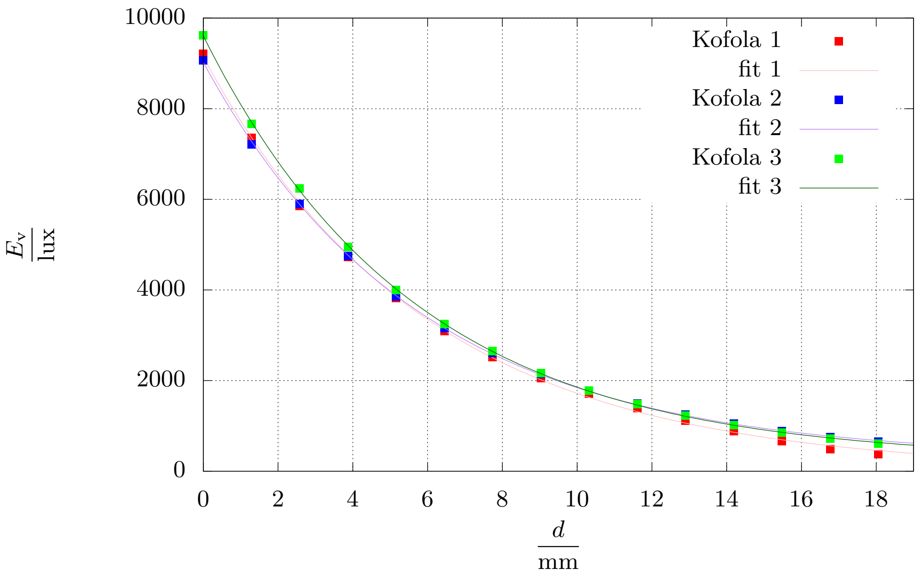

Fig. 2: Dependence of illuminance on surface height of Kofola.

We observe that the errors of the parameters are at least one, but rather two orders of magnitude lower than the parameters themselves. It means that we chose a suitable function to fit the measured data; this function captures the dependence of light intensity on the height of the liquid in the container very well. This is also clearly visible in the graphs 2 and 3. In converting the 30 g of the beverage to its corresponding height in the box, we neglected the box

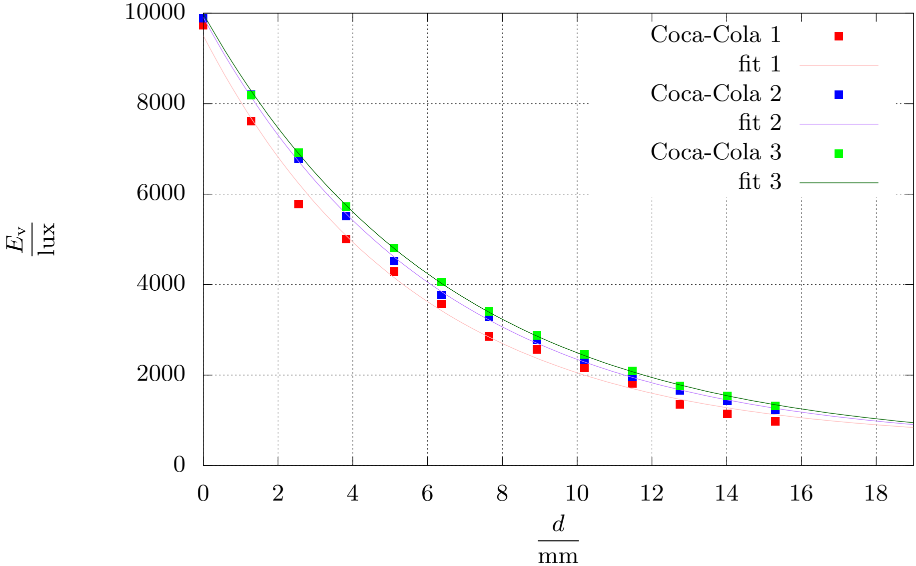

Fig. 3: Dependence of illuminance on surface height of Coca-Cola .

dimensions errors and the densities of drinks, as they are less than one percent. The uncertainty of the device, i.e., the uncertainty in determining illuminance is unknown. However, the device measures with an accuracy of units of lux, while our measurements are in the range of hundreds to thousands, so we can assume that the device error will not be significant.

The fitting errors of the parameter α are very small, so we can disregard them. Since we performed three measurements, we obtained the resulting value of the parameter α for each drink as the average of all three values, and we also calculated the standard deviation. We obtain

$$\alpha_{\text{Kofola}} = (0.172 \pm 0.002)\,\text{mm}^{-1}\,.$$

Before calculating the value of this parameter for Coca-Cola, let us pause. The graph for Coca-Cola shows that the measured data points from the first measurement are significantly lower than the other two measurements. During the first measurement, many small bubbles formed at the bottom, likely affecting the light passage to the sensor. Also, the value of α for the first measurement is significantly higher than for the other two. In them, no bubbles appeared. For calculating the average and standard deviation, we will use data only from the second and third measurements of Coca-Cola. We obtain

$$\alpha_{\text{Coca-Cola}} = (0.157 \pm 0.004)\,\text{mm}^{-1}\,.$$

The absorption coefficient of Kofola is higher than that of Coca-Cola, suggesting that Kofola allows less light to pass through. We even observed it with the naked eye during the measurement. Thus, the amount of light passing through Kofola decreases by a factor of e over a length of $1/\alpha_{\text{Kofola}} = 5.8\,\text{mm}$, and similarly, for Coca-Cola, it is $6.4\,\text{mm}$

Discussion

The home-based experiment faced limitations in available equipment. Precision in determining the height of the liquid within the box posed the most challenging task, but the chosen procedure proved to be the most accurate. If we had placed a mark with height in the box, reading from the scale could be distorted by capillary and other phenomena. Since we proceeded with relatively small amounts of drink ($30\,\mathrm{g}$), we did not have a suitable object to create such a small volume. Employing a greater volume would result in a condition where the light intensity undergoes minimal changes even with fewer steps. Due to the low resolution of scales, we are unsure about the precision of the weight-volume calibration. Therefore, we measured more points and fitted the data with a line. Using a pipette or a precise graduated cylinder would be more appropriate.

Unfortunately, we could not find information about the accuracy of the measured light intensity values. However, the relative error of individual measurements is considered negligible due to averaging over several seconds (see above).

Given the minuscule errors in the parameters obtained by fitting the values of light intensities with the presented function, we can conclude that we chose the correct function. We can make the same conclusion by looking at the graphs, where all data points are near the considered function. Thus, we verified our theoretical derivation experimentally. Although we neglected the dependence of the absorption coefficient on the wavelength, we concluded that it was justifiable. We saw that both drinks changed the color of the passing light, but it did not significantly impact our considerations.

We did not include the first measurement of Coca-Cola in the overall absorption coefficient due to unreliable results. The formation of bubbles reduced the passage of light, seemingly increasing the absorption coefficient α. However, the bubbles gradually disappeared during the first measurement, and the other two can be considered accurate. Conducting one more measurement to have a comparable result with Kofola would be appropriate.

The absorption coefficient of Kofola was higher but not significantly. The light intensity when passing through both drinks drops to a tenth already at about $1.5\,\mathrm{cm}$, which is why the bottle of this drink seems opaque or why we cannot see the bottom in a full glass.

Conclusion

We measured the dependence of light intensity on the thickness of two types of cola drinks – Kofola and Coca-Cola. Due to the available experimental equipment, we also had to determine the density of both drinks. From the errors of the parameters obtained by fitting and from a simple look at the graphs, we can state that we fitted the data with a suitable function and that the decrease is exponential with thickness. We determined the absorption coefficients as $\alpha_{\text{Kofola}} = (0.172 \pm 0.002)\,\mathrm{mm}^{-1}$ and $\alpha_{\text{Coca-Cola}} = (0.157 \pm 0.004)\,\mathrm{mm}^{-1}$.

Jaroslav Herman
jardah@fykos.org

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