Problem I.P ... rocket

10 points; průměr 3,24; řešilo 74 studentů

Using current technology, how much fuel would it take to carry an object of mass $m = 1\,\mathrm{kg}$ into low Earth orbit?

The leprechaun wanted to save on rocket fuel.

Introduction

The European Space Agency (ESA) specifies the Low Earth Orbit (LEO) as a range of 160 km – 1 000 km (other sources indicate up to 2 000 km corresponding to an orbital period of up to 128 min). Thus, for the purposes of this problem, we shall only consider cargo transport methods that successfully transported an arbitrarily large object into this orbit in the past. By this requirement, we have limited the range of transport vehicles to only a few types of rockets, as other methods have not yet been put into practice.

We assume that our object of mass $m_\mathrm{p} = 1\,\mathrm{kg}$ is the only object we need to carry, so we seek a rocket with a payload capacity of at least 1 kg while being as light as possible. The Japanese SS-520 rocket suits these requirements the best.

Orbit

We aim to launch the rocket with the object into an orbit, which we choose to be a circular orbit at $h = 160\,\mathrm{km}$ above the Earth's surface. If our objective is for the object (of a similar weight to the CubeSats) to remain in orbit without the need for adjustments for the typical duration of CubeSat missions (which usually last from weeks to a couple of months) and considering that CubeSats are typically deployed at altitudes around 400 km above the Earth's surface to mitigate atmospheric drag, we would need to position the object at a higher altitude. However, to minimize the required amount of fuel in this scenario, a height of $h = 160\,\mathrm{km}$ is deemed sufficient. We can calculate the required orbital velocity for this case by using the formula for the first cosmic velocity

$$v_\mathrm{k} = \sqrt{\frac{GM}{r}}\,,$$

where G is the gravitational constant, M is the mass of the Earth, and r is the radius of the objest's orbit. In our case $r = r_\mathrm{Z} + h = 6\,538\,\mathrm{km}$, where $r_\mathrm{Z} = 6\,378\,\mathrm{km}$ is the mean radius of the Earth. We get the velocity $v_\mathrm{k} = 7.81\,\mathrm{km{\cdot}s^{-1}}$.

We can consider Earth's rotation to lower the initial velocity needed. By aligning the rocket's direction with the Earth's rotation, we only need to provide a velocity equivalent to the difference between the escape velocity and the Earth's rotational velocity at the launch site. Therefore, we are looking for a cosmodrome close to the equator to maximize the initial speed.

The Guiana Space Centre in French Guiana is an ideal candidate at $5°3'$ N. An object on Earth at this latitude moves on a circle with radius $R_\mathrm{o} = R_\mathrm{W} \cdot \cos\varphi \doteq 6\,353\,\mathrm{km}$ with the period $T = 1\,\mathrm{day} = 86\,400\,\mathrm{s}$. Due to Earth's rotation, the rocket has a velocity $v_\mathrm{Z} \doteq 0.46\,\mathrm{km{\cdot}s^{-1}}$ at liftoff. Hence, we only need to accelerate the rocket (tangentially to the Earth) to a velocity of $v_\mathrm{t} = v_\mathrm{k} - v_\mathrm{Z} = 7.35\,\mathrm{km{\cdot}s^{-1}}$.

Gravitational pull and drag

The magnitude of the gravitational force as a function of the height h and the mass of the rocket m is

$$F_\mathrm{G} = \frac{GMm}{(r_\mathrm{Z}+h)^2}\,.$$

The gravitational acceleration at $h = 160\,\mathrm{km}$ is equal to $a \doteq 9.32\,\mathrm{m{\cdot}s^{-2}}$.

As the rocket travels through the atmosphere, it encounters a drag force described by the formula:

$$F_\mathrm{O} = \frac{1}{2} C_\mathrm{r} S \rho v^2\,,$$

where C_r represents the drag coefficient determined by the rocket's shape, S denotes the area of the rocket encountering the drag force (its cross-section in our case), ρ is the density of the atmosphere, and v is the rocket's velocity. Let us consider $C_\mathrm{r} = 0.3$ (coefficient for the shape of a bullet) and the diameter of the rocket's cross-section as $d = 0.52\,\mathrm{m}$, from which we get $S = \pi d^2/4 \doteq 0.21\,\mathrm{m^2}$ for its surface.

The atmosphere's density can by approximated by an exponential relation[1]

$$\rho \approx \rho_0 \mathrm{e}^{\frac{h}{H_n}}\,,$$

where ρ_0 is the density of the atmosphere at sea level, h is the height above the surface, and H_n is a constant defined as

$$H_n = \left(\frac{gM}{RT_0} - \frac{L}{T_0}\right)^{-1}\,,$$

where g is the gravitational acceleration at sea level, M is the molar mass of the atmosphere, R is the molar gas constant, T_0 is the atmosphere's temperature at sea level, and L is the rate of the temperature drop of the atmosphere. We obtain $H_n \doteq 10.4\,\mathrm{km}$.

Attributes of the rocket

The SS-520 rocket consists of two or three stages. Notably, the SS-520-5, the only variant in this series that has already successfully delivered an object to LEO, utilizes a three-stage configuration. However, as the third stage primarily serves for minor trajectory adjustments, we will focus only on the first two stages in our example. The mass of the entire rocket (excluding cargo and fuel) is approximately $m_\mathrm{r} = 580\,\mathrm{kg}$ (the often quoted mass of $m_\mathrm{c} = 2\,579\,\mathrm{kg}$ is the fully loaded rocket, including fuel), with the first stage weighing approximately $m_\mathrm{I} = 540\,\mathrm{kg}$, and the second and third stages together $m_\mathrm{II} = 40\,\mathrm{kg}$.[2] Both the first and second stages use a solid-fuel engine providing a thrust of $F_m = 185\,\mathrm{kN}$, with a specific impulse of $I_\mathrm{sp} = 265\,\mathrm{s}$.[3] Based on these values, we can calculate the exhaust velocity of the fumes relative to the rocket

$$u = I_\mathrm{sp} g \doteq 2\,600\,\mathrm{m{\cdot}s^{-1}}\,,$$

and the mass flow rate of the fuel using the equation

$$Q_\mathrm{m} = \dot{m} = \frac{F_\mathrm{m}}{I_\mathrm{sp} g}\,.$$

[1] Density of air, available at `https://en.wikipedia.org/wiki/Density_of_air`

[2] World's Smallest Launch Vehicle Ready for Second Attempt at Reaching Orbit, available at `https://spaceflight101.com/ss-520-5-launch-preview/`

[3] SS-520 Nano satellite launcher and its flight result, available at `https://digitalcommons.usu.edu/cgi/viewcontent.cgiarticle=4120&context=smallsat`

We get $Q_\mathrm{m} \doteq 71.2\,\mathrm{kg\cdot s^{-1}}$.

Flight model

For simplicity, we will divide the rocket flight into two steps. One is the flight of the entire rocket (including the first stage) vertically up from the Earth. In the second phase, the first stage is separated, and the rocket is accelerated to achieve the required orbital velocity, enabling it to travel tangentially in orbit.

In addition to its velocity, the rocket changes its mass during the flight as the fuel is used and expelled. First, we will calculate the amount of fuel needed to deliver the circular velocity using the second stage of the rocket. For this, we can use the Tsiolkovsky rocket equation since no external forces are acting on the body in the tangential direction from the Earth at the height of the orbit.

Since there are (non-constant) external forces acting on the rocket during the flight, we can no longer use the Tsiolkovsky rocket equation to calculate the amount of fuel needed for the first part of the flight (the upward flight). Thus, we will use the more general Meshchersky's equation for rocket motion.

Accelerating at the first cosmic velocity

We denote the mass at the beginning of the maneuver $m_\mathrm{start} = m_\mathrm{II} + m_\mathrm{p} + m_\mathrm{fuel,2}$, for the mass at the end holds $m_\mathrm{end} = m_\mathrm{II} + m_\mathrm{p} = 41\,\mathrm{kg}$. The rocket must change its velocity throughout the maneuver in a tangential direction by $\Delta v = v_\mathrm{t} = 7.35\,\mathrm{km\cdot s^{-1}}$, Tsiolkovsky rocket equation states

$$m_\mathrm{start} = m_\mathrm{end} \mathrm{e}^{\Delta v / u}\,,$$

where u is the exhaust velocity. By substituting the known values, we get the initial mass of this phase of flight $m_\mathrm{start} \doteq 693\,\mathrm{kg}$ and, therefore, the mass of the fuel in the second stage $m_\mathrm{fuel,2} = 652\,\mathrm{kg}$.

Equation of motion

For the model of the rocket's upward motion, we start with Meshchersky's equation for the rocket motion

$$\dot{\mathbf{p}} = m\ddot{\mathbf{x}} + \dot{m}\mathbf{u} = \mathbf{F}\,,$$

where $\dot{m}$ is the fuel rate, $\mathbf{u}$ is the exhaust speed relative to the rocket, $\mathbf{p}$ is the rocket's momentum, m is the current mass of the rocket, and $\mathbf{F}$ is the resultant force acting on the rocket (gravitational and drag). By substituting the individual forces acting upon the rocket we get

$$m\ddot{x} = \dot{m}u - \frac{GMm}{(r_\mathrm{Z} + x)^2} - \frac{1}{2} C_\mathrm{r} S \rho_0 \mathrm{e}^{-\frac{x}{H_n}} v^2\,,$$

after substituting $v = \dot{x}$ and dividing the whole equation by m we obtain

$$a = \ddot{x} = \frac{\dot{m}}{m} u - \frac{GM}{(r_\mathrm{Z} + x)^2} - \frac{1}{2m} C_\mathrm{r} S \rho_0 \mathrm{e}^{-\frac{x}{H_n}} \dot{x}^2\,.$$

This equation has no analytical solution, but we can attempt to solve it numerically using a computer simulation.

Simulation of the first phase of flight

To calculate the modified equation of motion, we use Euler's method for solving ordinary differential equations. We will increase the time in suitably small steps (in our case, for example, $\Delta t = 0.1\,\mathrm{s}$) and observe how the position and velocity of the rocket change during this time. Thus, we get an equation for the position (height above the surface) of the rocket

$$x_{t+\Delta t} = x_t + \Delta t v_t \,,$$

where x_t and v_t represent the position or the velocity of the rocket at time t respectively, and $x_{t+\Delta t}$ is the position of the rocket Δt later.

The second equation is for velocity

$$\dot{x}_{t+\Delta t} = v_{t+\Delta t} = v_t + \Delta t a_t \,,$$

where a_t is the acceleration of the rocket at time t. By substituting for acceleration, we get

$$\dot{x}_{t+\Delta t} = v_{t+\Delta t} = v_t + \Delta t \left(\frac{\dot{m}}{m} u - \frac{GM}{(r_\mathrm{Z} + x_t)^2} - \frac{1}{2m} C_\mathrm{r} S \rho_0 \mathrm{e}^{-\frac{x_t}{H_n}} {\dot{x}_t}^2 \right) .$$

As the initial conditions, we choose $t = 0\,\mathrm{s}$, $x = 0\,\mathrm{m}$ (corresponding to the Earth's surface) and $\dot{x} = 0\,\mathrm{m{\cdot}s^{-1}}$ (the rocket has initial velocity equal to zero). At the end of the first phase of flight, the remaining mass of the rocket must be equal to $m_0 = m_I + m_\mathrm{II} + m_\mathrm{p} + m_\mathrm{fuel,2} = 1\,233\,\mathrm{kg}$.

Since we do not know the total initial mass of the rocket corresponding to $m = m_0 + m_\mathrm{fuel,1}$, where $m_\mathrm{fuel,1}$ is the mass of the fuel in the first stage of the rocket, we will run the simulation consecutively for different amounts of fuel. We choose $m_\mathrm{fuel,1} = 0\,\mathrm{kg}$ as the lower estimate, and evaluate the upper estimate by ensuring that the rocket can lift off from the surface under the initial conditions, i.e., the acceleration is $\ddot{x} \geq 0\,\mathrm{m{\cdot}s^{-2}}$. By substituting the initial conditions into the equation of motion, we arrive at an upper estimate of the mass

$$m_{fuel,1} \leq \frac{r_\mathrm{Z}^2 F_\mathrm{m}}{GM} - m_0 \doteq 17\,660\,\mathrm{kg} \,.$$

At this interval, we simulate each whole unit of kilogram until the rocket reaches the desired altitude (at which point we have found the minimal mass). A sample of the code used (in Python) is in the attached file.[4]

Result

Based on the simulation, the mass of the first stage fuel came out as $m_\mathrm{fuel,1} = 1\,275\,\mathrm{kg}$. Therefore, the total amount of fuel needed to carry the object is $m_\mathrm{fuel,1} + m_\mathrm{fuel,2} \doteq 1\,927\,\mathrm{kg}$. The mass of the entire rocket with cargo and fuel is $m = 2\,508\,\mathrm{kg}$. The traceable total mass of the

[4] `https://drive.google.com/file/d/1SbvIW9X-0cKGT39NKBjj--fl8F7HONBj/view`

SS-520-5 rocket (when flown on 3/2/2018) was $m_\mathrm{c} = 2\,579\,\mathrm{kg}$, which is reasonably consistent with the calculated value.

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