Problem I.1 ... Moby Dick

3 points; průměr 1,66; řešilo 122 studentů

Some species, such as cetaceans, navigate by echolocation. Let us assume that a cetacean emits a sound signal through a larynx located precisely between the ears at a distance a. Consider a submarine is moving at the same depth as the whale. The sound bounces off the submarine and arrives at the closer ear of the whale at time t from the moment of transmission. If the time delay between the sound picked up by the right and the left ear is Δt, what is the distance and direction of the submarine? *The whale expedition got a bit out of Radka's hands.*

Firstly, let us sketch a picture 1 of the situation. Considering the submarine and the whale are at the same depth, we can think of the problem as a two-dimensional problem. We denote the distance of the whale's ears by a and the distance from the submarine to the closer ear as s. We want to calculate the distance to the submarine l, defined as the distance from the submarine to the whale's larynx. Furthermore, we will define an acute angle α between the axis of the whale and the line connecting the submarine and the whale's larynx; see the picture below.

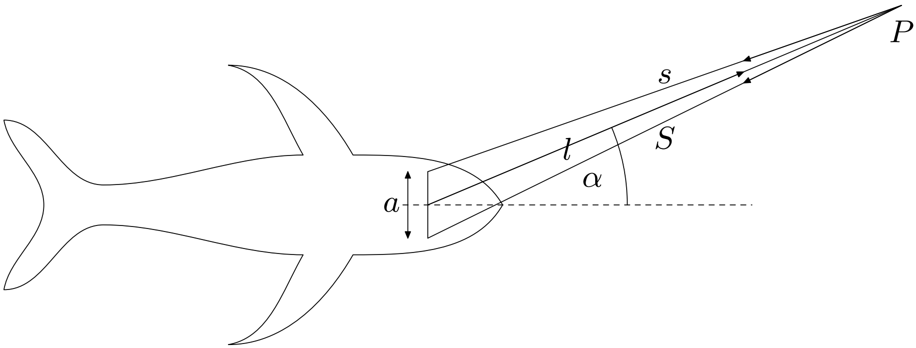

Fig. 1: The picture of the situation. The letter P denotes the location of the submarine. Arrows indicate the direction of the sound propagation.

Let us denote the speed of sound in the water by v. For the total signal propagation time t we can write

$$l + s = vt\,.$$

Similarly, for the difference between the right and the left ear Δ, we can write

$$\Delta = v\Delta t\,.$$

Notice that the triangle inequality implies $v\Delta t \leq a$. We can express the distance from the submarine to the farther ear as

$$S = s + \Delta\,.$$

Now, by using the law of cosines for the triangle: the larynx – the closer ear – the submarine we will get

$$s^2 = \left(\frac{a}{2}\right)^2 + l^2 - 2\frac{a}{2}l\cos(90^\circ - \alpha)\,.$$

After simplification

$$s^2 = \frac{a^2}{4} + l^2 - al\sin(\alpha)\,.$$

We can also use the law of cosines for the second triangle: larynx – farther ear – submarine

$$S^2 = \left(\frac{a}{2}\right)^2 + l^2 - 2\frac{a}{2}l\cos(90^\circ + \alpha)\,,$$

Simplifying to

$$s^2 + 2s\Delta + \Delta^2 = \frac{a^2}{4} + l^2 + al\sin(\alpha)\,.$$

By adding these two equations, we get

$$2s^2 + 2s\Delta + \Delta^2 = 2\frac{a^2}{4} + 2l^2$$

and by substituting t and Δt for s and Δ we get

$$2v^2t^2 + 2l^2 - 4vtl + 2(vt - l)v\Delta t + v^2(\Delta t)^2 = \frac{a^2}{2} + 2l^2\,.$$

Notice quadratic terms with l will cancel out. From the obtained linear equation, we can easily express the distance of the submarine l as

$$l = \frac{v^2t^2 - \frac{a^2}{4} + vtv\Delta t + \frac{v^2(\Delta t)^2}{2}}{2vt + v\Delta t}$$

or (by factoring out $vt/2$)

$$l = \frac{vt}{2}\,\frac{1 - \frac{a^2}{4v^2t^2} + \frac{\Delta t}{t} + \frac{(\Delta t)^2}{2t^2}}{1 + \frac{\Delta t}{2t}}\,.$$

By subtracting the two law of cosines equations, we get

$$2al\sin\alpha = 2s\Delta + \Delta^2 = 2(vt - l)v\Delta t + v^2\Delta t^2\,,$$

from which we can easily express

$$\sin\alpha = \frac{2(vt - l)v\Delta t + v^2\Delta t^2}{2al} = \frac{v\Delta t}{a}\,\frac{vt - l + \frac{v\Delta t}{2}}{l} = \frac{v\Delta t}{a}\left[\frac{vt}{l}\left(1 + \frac{\Delta t}{2t}\right) - 1\right].$$

After substituting for l and some straightforward algebraic simplifications, we get the final expression

$$\sin\alpha = \frac{v\Delta t}{a}\,\frac{1 + \frac{a^2}{4v^2t^2} + \frac{\Delta t}{t}}{1 - \frac{a^2}{4v^2t^2} + \frac{\Delta t}{t} + \frac{(\Delta t)^2}{2t^2}}\,.$$

It is particularly interesting to examine the limit case when

$$\frac{a}{vt} = \frac{a}{s + l} \ll 1\,,$$

which also can be interpreted as when the distance of the submarine is much larger than the width of the whale (what is typically expected). This also implies (by using the triangle inequality)

$$\frac{\Delta t}{t} \leq \frac{a}{vt} \ll 1\,,$$

which means that the delay of the signal Δt is negligible compared to the total signal propagation time t. Assuming this limit, we can simplify more, giving us final results for l and α

$$l \approx s = \frac{vt}{2}$$

and

$$\sin\alpha \approx \frac{v\Delta t}{a}\,.$$

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