Problem VI.E . . . ripples

13 points; průměr 7,76; řešilo 33 studentů

Build an apparatus that can measure the smallest possible ripples on the surface of the liquid. You can choose the container yourself – it can be a cup, a bottle, or something else. Thoroughly describe and take a picture of the whole apparatus. Determine the minimum amplitude you are able to measure. *Karel was staring into space… he was writing his dissertation thesis.*

Introduction

The goal of the problem is to measure the smallest possible waves on the water surface. Thus, we prioritize the need to get the best resolution possible over accuracy. At the same time, we need to measure continuously, so that we can detect the gradually decreasing amplitude of the waves. Finally, we need to determine if the measured data still correspond to the waves and when we are only measuring noise.

Procedure 1: float

Measurement

The first method we used was to directly measure the water level by recording it on video, using a small float for better visualization. We used glass and larger kettle as containers to test two different settings varying in size. Moreover, we used a ball of allspice as a float. We filmed the whole process on a smartphone camera and processed the video in *Tracker*. One point on the underside of the float and one point of contact with the water were used for observation, as the float's center was difficult to pinpoint. You can see a sample of the video processing in figure 1.

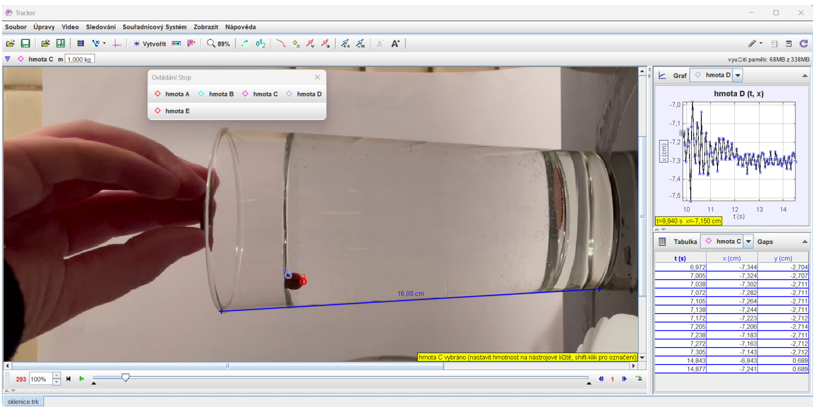
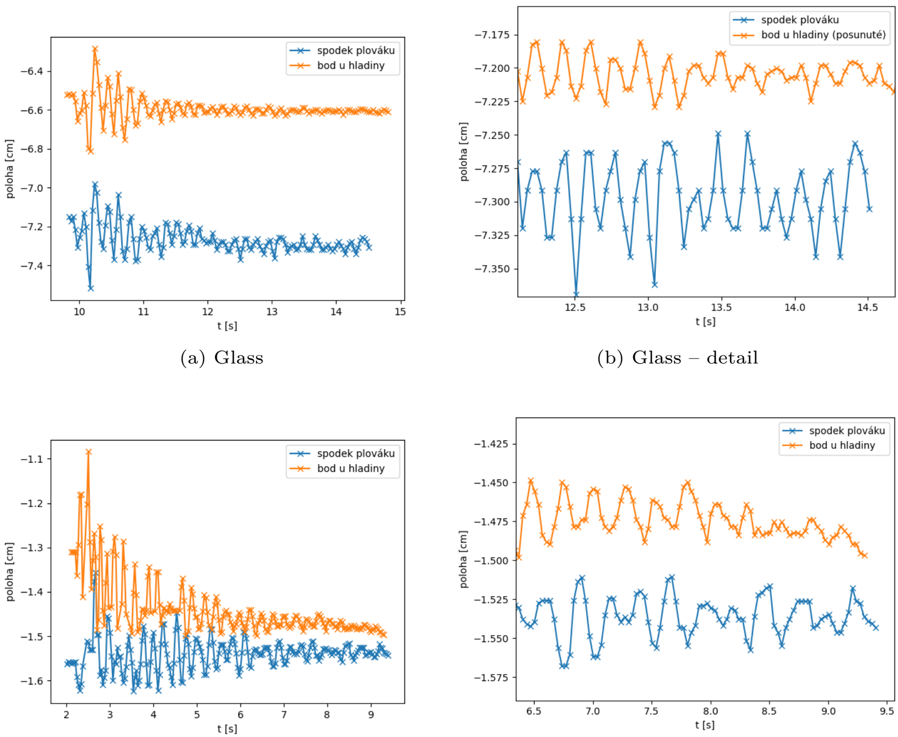


Fig. 1: Sample from the data processing of the measurements using a float.

The plot of the position of the monitored points versus time for the two containers is shown in the graphs 2.

For both containers and both measurement points, we can distinguish the smallest amplitudes of the order of 0.2 mm. We additionally measured a distance of 20 pixels in the Tracker



(a) Glass

(b) Glass – detail

(c) Kettle

(d) Kettle – detail

Fig. 2: Dependence of the displacement of a float on time in both containers.

program, from where we determined the center distance of two adjacent pixels to be approximately 0.2 mm, which is the same as the smallest measured amplitude.

Discussion From the ratio of the smallest measured amplitude and the theoretical accuracy limit (the center distance between two pixels), we can see that we have reached the technical limit of the apparatus constructed in this way because the determination of the deflections of the float position to a higher accuracy than one pixel on the record is very problematic. We can see from the figure 2 that the measured positions of the bottom of the float in the kettle were in antiphase with the positions of the point at the surface, indicating that the float's inertia had been too great, and the float did not adapt to the changes immediately. For the glass, due to the different frequencies of the waves, this problem does not occur, but this phenomenon suggests that even a float this small can influence the waves significantly. We could have achieved a higher accuracy by capturing a smaller area, with one pixel showing a smaller actual distance.

Procedure 2: laser reflections

Theory

The second method leveraged the law of reflection of a laser beam from the water surface. In this measurement, we shine a laser at the water surface at a small angle from the perpendicular and observe the motion of the reflection on a distant shield e.g., the ceiling. When the water level is deflected by φ, the change in the angle of the reflected beam will be 2φ. We denote the distance of the surface to the ceiling as h and the deviation from the equilibrium position of the reflection of the beam as x. For small angles, we can write the following

$$2\varphi \approx \tan 2\varphi = \frac{x}{h}\,.$$

The disadvantage of this method is that we are measuring the tilt angle, not directly the amplitude, which we have to estimate from the angle.

To estimate the amplitude, we first take a closer look at the water waves in a container. Since the container is spatially confined, the induced wave will have repeated reflections off the wall until, after some time, a standing wave is generated with the parameters we shall attempt to estimate. The standing waves cannot have an arbitrary frequency and wavelength. Quite the opposite, their parameters depend on the dimensions of the container and the so-called mode, which is simply the number of waves that can fit into the space (the container). Furthermore, waves are damped, which is mainly due to the viscosity of the water. The damping is not as strong for all modes, but higher modes are damped faster. So, for our amplitude estimation, we will only consider the fundamental mode, treating the glass as the resonator with free ends, since the water level at the walls also fluctuates. We will then consider the simplest case in which a wave has a wavelength equal to twice the diameter of the glass d. The shape of the wave dependent on the coordinate x and time t can then be described by the equation

$$A(x,t) = A_0 \sin\left(2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right)\,,$$

where T is the period and λ is the wavelength. The slope of the surface φ, or the tangent of this angle, is given by differentiating with respect to the coordinate x

$$\tan\varphi = \frac{A_0 2\pi}{\lambda}\cos\left(2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right)\,.$$

If we are looking for the maximum of this function, we put the cosine equal to 1. We get the maximal amplitude as

$$\tan\varphi = \frac{A_0 2\pi}{\lambda} = \frac{A_0\pi}{d}\,.$$

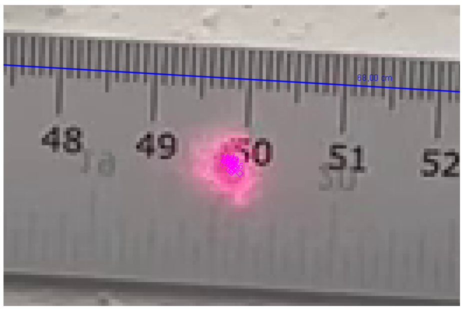
If we again consider the small angle approximation and the wavelength equal to twice the diameter of the glass (i.e., maximum oscillations in the middle), we can put the tangent of the angle equal to the angle in radians, so we can write

$$\frac{x}{2h} = \frac{A_0\pi}{d} \quad\Rightarrow\quad A_0 = \frac{xd}{2\pi h}\,, \tag{1}$$

which gives us an estimate of the maximum amplitude of the wave we are measuring. The wave then causes the beam to be deflected by x.

Measurement methods and results In our measurements, we used a red laser pointer that we fixed on the table in a holder and shone the laser on the water surface in a glass placed on the floor. We used the room's ceiling, onto which we taped the meter. We filmed the area around the equilibrium position of the reflection with a smartphone camera. The apparatus is shown in the figure 3a, an example of the reflection used for processing is in the figure 3b.

(a) Apparatus used



(b) Sample of measured data

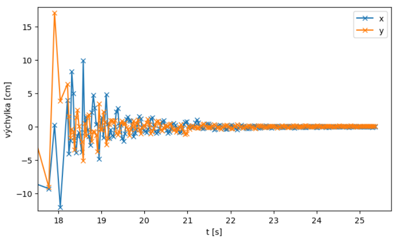
Fig. 3: Laser reflection measurement.

We measured the reflection positions in the Tracker software, which we calibrated using the displayed meter. The remaining parameters of the equipment were measured using a tape measure or callipers and are shown in table 1. We determined the measurement uncertainties as one-half of the smallest division, and in the case of height, the sum of the uncertainties of measurement of the height of the room 0.5 cm and the uncertainty of the measurement of the surface position, which, due to capillary effects near the container wall was estimated to be 1 mm.

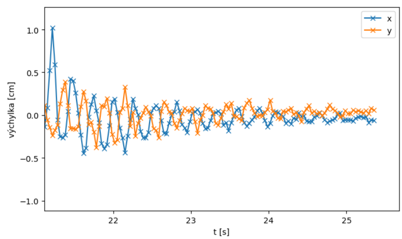
Tab. 1: Laser measurement parameters.

	glass	kettle
$\frac{d}{\mathrm{cm}}$	7,22±0,05	14,3±0,1
$\frac{h}{\mathrm{cm}}$	265,4±0,6	272,9±0,6

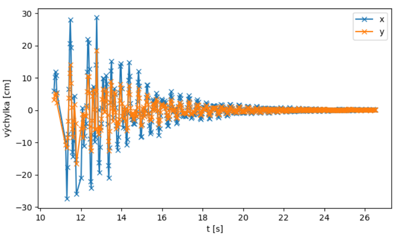
The measurements were again made with two containers, each with a different radius and surface height, so the distance from the surface to the ceiling was different as well. The measurements in the Tracker were the coordinates of the reflection positions on each image, and the equilibrium position determined as their average. The deviations from equilibrium in both coordinates and for both containers are plotted in figure 4.



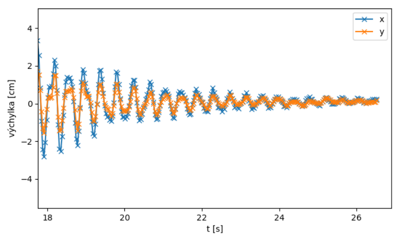
(a) Glass



(b) Glass – detail



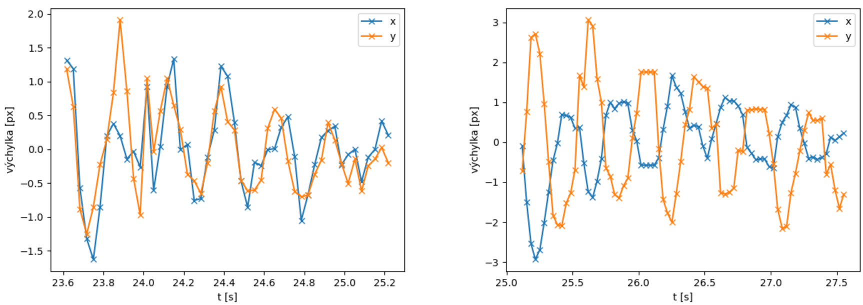
(c) Kettle



(d) Kettle – detail

Fig. 4: Dependence of laser deflection on time in both dimensions.

From the graphs, we see that we can identify waves up to amplitude 0.2 cm for the glass, while for the kettle, only up to 0.5 cm. However, during data processing focused on smaller amplitudes, we observed that the change in the position of the laser reflection is recognizable, but very difficult to determine reliably, as the reflection's size on the ceiling is almost half a centimeter. Therefore, we decided to use the *Python* program, into which we loaded the individual frames of the video. In these, we have calculated the position of the center of the reflection (using the intensity in the red channel as a weight), and we used these values as the reflection positions. The dependence of the reflection position on time is plotted in the graphs in the figure 5. Since the calibration from the Tracker no longer works here, the values of the deflections are given in pixels, and we used a new calibration given by the positions of the extreme points of the meter. This calibration shows that one pixel corresponds to 0.58 mm for the glass measurement and 0.39 mm for the kettle measurement. Since the uncertainty in the determination of the two extreme points is ±1 px and the distance of these points corresponds to tens of pixels, the uncertainty of the calibration is less than 1 %, and therefore negligible relative to the other errors.



(a) Glass (b) Kettle

Fig. 5: Detailed graph of deflections processed in Python.

Here we can already identify waves with an amplitude of $0.5\,\mathrm{px} \doteq 0.29\,\mathrm{mm}$ for the glass measurement and $1\,\mathrm{px} \doteq 0.39\,\mathrm{mm}$ for the kettle measurements. In fact, our measurements of waves are $\sqrt{2}$-times larger because the amplitude has a given magnitude in two perpendicular directions, so for the glass, we have $x = 0.41\,\mathrm{mm}$ and for the kettle $x = 0.55\,\mathrm{mm}$. Due to the inaccuracy of the amplitude determination, we estimate the uncertainty of the measurement to be 10 %, so this is more of an order of magnitude estimate, but that does not bother us too much in this case. After substituting into the formula (1), we get the wave amplitude for measurements in the glass $A_0 = (1.8 \pm 0.1)\ \mathrm{\mu m}$ and in the kettle $A_0 = (4.6 \pm 0.5)\ \mathrm{\mu m}$.

Discussion With this method, we got to an order of magnitude higher accuracy. We were able to identify waves of amplitudes of the order of micrometers. However, this method essentially measures the angle of inclination of the surface from which we only estimate the amplitude assuming a standing wave of a given wavelength and aiming the laser at its node (i.e., the point with the biggest inclination). The accuracy of the amplitude determination is thus limited (it is more of an order of magnitude estimate), but from this measurement, we can say with great confidence, whether or not the surface is rippling. The main problems with this measurement method are the imperfections of the laser beam, which is not exactly point-like, and its divergence, which causes the trace on the screen to have a diameter of about half a centimeter. We would need a laser of better quality with a smaller footprint to improve the sensitivity of the measurement. Other measurement uncertainties include inaccurate calibration of the deflection of the laser reflection measurement, either by inaccurate determination in the picture or by imperfect attachment of the tape measure to the ceiling, which was slightly bent. In addition, this measurement neglects the change in the distance of the surface from the shield, which is small relative to the size of the wave amplitude.

In this section, we have assumed that the wave is one-dimensional, which means we have not considered the circular shape of the glass. To take this into account, we would need to solve the wave equation in a circular space, which would involve Bessel functions with many different modes. Describing these functions is complicated, so the resulting equations must be solved numerically. The results only differ slightly from the plane wave approximation. Therefore, we have used an approximate procedure sufficient for our order of magnitude estimate. For those who want an exact description of waves on the water surface, we recommend referring to an undergraduate thesis on this subject.[1]

Discussion

Both measurements of the waves consisted of observing the image on the video as accurately as possible, whether it was the float in the first case or the laser reflection in the second. The accuracy of the measurement could be improved both by using a camera with higher spatial and temporal resolution or by scanning a smaller portion of the area. In the case of a laser, we could then project its reflection directly onto the camera chip, but we would have to use suitable filters to avoid damaging it. However, even when using a conventional phone, we got down to fractions of a millimeter in direct measurement and to units of micrometers while using a laser beam, which is very accurate.

For both measurements, we were primarily concerned with the estimation of the capabilities of the apparatus used rather than an accurate measurement of the wave in question, so individual amplitude measurements have an uncertainty of tens of percent. If we wanted to achieve even higher accuracies, we could use an interferometer that would allow us to measure with an accuracy comparable to the wavelength of the light used –the order of hundreds of nanometers. However, the construction of the interferometer would require special tools and a demanding tuning of the apparatus.

Conclusion

We used two methods to measure the amplitude of waves at the water surface. By directly measuring the amplitude using the camera recording, we were able to measure waves of approximately $0.2\,\mathrm{mm}$. On the other hand, when measuring the reflection of the laser off the water surface we were able to get up to waves of amplitude $(1.8 \pm 0.1)\,\mathrm{\mu m}$ for the glass measurements

[1] `https://is.muni.cz/th/mqi9u/Peckova_Vlny_na_vodni_hladine.pdf`

and (4.6 ± 0.5) μm for kettle measurements. Thus, the method of laser beam reflection allowed us to measure up to a hundred times smaller waves.

Kateřina Rosická
kacka@fykos.org

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