Problem VI.4 ... light faster than light

7 points; průměr 3,88; řešilo 40 studentů

There is a laser in the distance L from a large screen. Initially, the laser shines on the screen so that the distance from a laser spot on the screen to the laser is $R > L$. Then at the time $t = 0\,\mathrm{s}$, we begin to rotate the laser at a uniform angular speed ω. Consequently, the distance of the spot on the screen from the laser decreases to L and then increases back to R. What is the speed of this laser spot relative to the screen? Is it possible that the spot moves at a speed greater than the speed of light in a vacuum? Is there a limit, can it be infinite? How (qualitatively) does this speed depend on the spot's position on the screen? The whole apparatus is in a vacuum.

Marek J. wanted to verify statements about the apparent surpassing of the speed of light.

When solving the problem assignment, we will be interested in the cases where $L/c \gg 1\mathrm{s}$. Since the problem asks whether the speed of a point of light on a screen can exceed the speed of light in a vacuum, c, we will look at what values it takes for $L \to \infty$. The purpose of this solution in addition to describing the result, is to provide and use several approaches to problems that will hopefully help you successfully calculate other similar assignments.

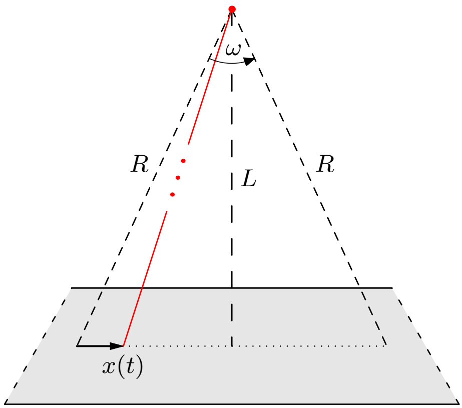

Fig. 1: Situation sketch. $x(t)$ marks the position of the light point at time t.

The geometry of the thought experiment consists of an equilateral triangle with the beginning of a beam at one vertex, with two sides of length R, while the third side consists of a portion of the screen.

The first advice for getting to the solution, in addition to understanding and one's own reformulation of the problem, is **simplifying the problem and solving it that way**.

We will leave the start of the experiment unchanged, but instead of an evenly rotating laser, let's consider two lasers: one identical to the laser from the beginning of the experiment and the other turned off, pointing perpendicular to the screen – side L. We begin the experiment by turning the first laser off and the second on.

If we placed the screen close to the lasers (which corresponds to everyday life), we would observe an instantaneous extinction of the first point of light and the simultaneous illumination of the second point of light. However, when $L/c \gg 1\,\mathrm{s}$ holds, then if the first laser is turned off and the second laser is turned on, the first point will still be visible on the screen, then the second point of light will appear, and just then will the first point go out. What happens is strange, as if the second laser was lit before the first laser was turned off. Let's calculate the times exactly, and hopefully, it will become clear what is happening.

The finite speed of light implies that it takes some time for light to reach the screen, similar to how the information about the turning-off of the first laser does not manifest on the screen instantaneously (the essence of the relativity of the present). Given times in the order above can be calculated as

$$t_1 = \frac{L}{c}\,,$$
$$t_2 = \frac{R}{c}\,.$$

Since, indeed, $R > L$, then $t_2 > t_1$, and thus the information about the second laser being turned-on arrives at the screen before the information about turning off the first laser. The effect of the next delay, caused by the lasers bouncing off the screen to the observer near the screen is ignored, as it would only exaggerate the observed phenomena.

We can see that the superluminal velocity mentioned above will be related to the changing length of the laser beam. We could also imagine the same effect (in the first approximation) by calculating the speed of the point of light using angular velocity ω as $v = l\omega$, where l is the actual beam length(i.e., the classical circumferential velocity) and at $l \to \infty$ v also goes to infinity. Is this really the case, however? Let's finally get to the solution of the problem from the assignment.

We now see that the behavior of the laser spot velocity on the screen is symmetrical to the perpendicular drawn from the vertex of the triangle to the screen, i.e., according to the length L defined in the problem. Here comes the second piece of advice for solving the problem: **find symmetry in the assignment and use it to simplify it**. In this case, it is sufficient to consider only one half of the triangle. Let us consider the beginning of the experiment at the point of incidence of the laser beam with length L and the end of the experiment at the point of incidence of the laser beam with length R.

Another important step is the *selection of the coordinate system.* In essence, it should capture the symmetry of the problem and save us from computing, whether by simplifying expressions or, for example, not having to keep adding constant terms.

In our case, we are well served by the coordinate x, which is identical to the position on the screen, and $x = 0$ marks the beginning of the experiment (the incident laser beam of length L). Similarly, the time t and the angle φ start at zero. For the angle, we can also write $\varphi = \omega t$. **Clearly defining the quantities associated with the task and awareness of the relationships between them** can often help in obtaining the solution.

We denote the position of a point of light as $x(t)$, and by using the Pythagorean theorem, we get:

$$x(t) = \sqrt{l^2(t) - L^2}\,, \tag{1}$$

where $l(t)$ is the trajectory length of the currently emitted laser photon, ending at the screen (consider that the beam, in this case, is not a straight line) for which (2) holds

$$l(t) = \frac{L}{\cos(\omega t)}\,. \tag{2}$$

If we did not consider any delay of the information about the laser motion, we would, by substituting the relation for (2) into (1) and then deriving $x(t)$ by time, get the dependence of the velocity of the point of light on time $v(t)$. However, we need to consider the delay. Thus, for $x(t)$, after substituting for $l(t)$, the following holds

$$x(t) = L\tan(\omega t)\,. \tag{3}$$

Let's get down to calculating the transmission delay of the motion of the laser beam. It depends on the length of the beam when it is sent out by the laser. The delay time is given by the photon's trajectory and divided by the speed of light. Let us, therefore, denote the time with a given delay with which the information is initially received by the imaginary observer at the shadow as t':

$$t' = t - \frac{l(t')}{c} = t - \frac{L}{c\cos(\omega t')}\,. \tag{4}$$

One might question whether l should not be dependent on time t, but the screen is reached by beams emitted from the past, i.e., at time t, the screen gets hit by a beam sent at time t' (delayed time), so the beam passing through the path $l(t')$ hits the screen (since the trajectories are already determined when the laser is pointed).

For the delayed time t' we thus get the transcendental equation, and therefore we cannot express it directly. However, it can be found numerically for given parameters. This does not bother us, since we can answer the questions in the problem with equations containing the "shifted" time "t'" since it does not change the behavior of the velocity (its values, maximum, and the like).

The fictional observer at the screen will see the beams or luminous points on the screen with a delay expressed in terms of (4), and we can express their position simply by substituting the relation for t' into the equation for $x(t)$, getting us

$$x' \equiv x(t') = L\tan\left[\omega\left(t - \frac{L}{c\cos(\omega t')}\right)\right]\,. \tag{5}$$

Deriving the relation (5) with respect to time t, gets us our desired dependence $v(t)$ as

$$v(t'(t)) = \frac{\omega L c}{c\cos^2(\omega t') + \omega L\sin(\omega t')}\,, \tag{6}$$

where we have used standard derivatives, using the relations above, or the properties of the goniometric tangent function. From equation (4), we computed

$$\frac{\mathrm{d}t'}{\mathrm{d}t} = \frac{c\cos(\omega t')}{c\cos(\omega t') + \omega L\tan(\omega t')}\,.$$

The relation (6) describes the velocity of a point of light relative to the screen for times $t' \geq 0$ (the point of light was stationary before that).

We can now answer all the remaining questions in the assignment.

For the angle φ, we consider positive values smaller than the right angle, thus, no problems will arise when examining the behavior for $L \to \infty$ while the relations containing the angle are non-zero and finite. In our following discussion, we can therefore ignore them.

Angular velocity ω and the speed of light c are similar. We can see that the numerator of the expression (6) contains L with the order of the denominator, and thus, for $L \to \infty$, the velocity is "modulated" by the sine in the denominator and approaching infinity in certain sections of the screen. Of course, we are not talking about actual infinity; in that case, the laser would never even reach the screen. The point is that we get an arbitrarily large velocity.

All that remains is a qualitative description of the velocity of the point of light on the screen in the thought experiment from the assignment. We see that initially, the light point will remain on the screen in the position corresponding to the beam R (the beginning of the experiment) after that, when the information about the laser's motion reaches the screen, the point starts to move towards the center of its trajectory, determined by the beam with length L, when a given velocity increases on the way to the center ($l(t)$ shortens). It reaches its maximum around the center, and then slows down to a second point determined by a second beam with a length R.

And now, the final question for the reader: why does this not contradict the special theory of relativity?

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