Problem V.5 ... xenon was wandering

8 points; průměr 3,15; řešilo 33 studentů

A once positively ionized xenon atom flew out from the center of a large cylindrical coil with velocity $v = 7\,\mathrm{m{\cdot}s^{-1}}$ and began to move through a homogeneous magnetic field, which is in a plane perpendicular to the magnetic lines of force. At a certain point the coil is disconnect from the source, thus its induction begins to decrease exponentially according to the following equation $B(t) = B_0 \mathrm{e}^{-\Omega t}$, in which $B_0 = 1.1 \cdot 10^{-4}\,\mathrm{T}$ and $\Omega = 600\,\mathrm{s^{-1}}$. What is the deviation from the initial direction after the atom is stabilized?

Vojta spent several hours thinking about a reasonable problem assignment with a clever solution, but ultimately, it ended horrendously. And he has yet to see the solution.

First, we need to recognize which forces are acting on the atom. Of course, there is the magnetic force caused by the presence of a magnetic field. But the magnetic field changes with time, so an electric field must also act on the electron. The Maxwell-Faraday equation for the circular region of radius r simplifies due to the symmetry of the problem to

$$\frac{\mathrm{d}B}{\mathrm{d}t}\pi r^2 = \frac{\mathrm{d}(\mathbf{B}\cdot\mathbf{S})}{\mathrm{d}t} = \frac{\mathrm{d}\Phi}{\mathrm{d}t} = \oint \mathbf{E}\,\mathrm{d}\mathbf{s} = E 2\pi r \quad \Rightarrow \quad E = \frac{r}{2}\frac{\mathrm{d}B}{\mathrm{d}t}\,.$$

Even though we have found the magnitude of the electric intensity vector, we still have to figure out its orientation. We can also express the previous equation in a differential form

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,.$$

The curl operator $\nabla\times$ is the cross product of the operator ∇ (which contains the partial derivatives by each coordinate) with the vector as argument (which in our case is the electric field intensity). For example, for the z-axis component of the resulting vector we have $\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}$.

Let us orient the coordinate system so that the z-axis points in the direction of the magnetic induction and is identical to the axis of symmetry of the cylinder. The axes x and y then lie in a plane perpendicular to this axis. Let the origin lie at the point from which the atom comes out, and let the x-axis point to the direction of its velocity. Then the magnetic induction vector has the form $\mathbf{B} = B_0 \mathrm{e}^{-\Omega t}\,(0,0,1)^\mathsf{T}$. Therefore, the vector produced by the curl operator on the electric intensity must only have the third component. You can easily check that the electric intensity vector

$$\mathbf{E} = \frac{B_0\Omega}{2}\mathrm{e}^{-\Omega t}\begin{pmatrix}-y\\ x\\ 0\end{pmatrix}$$

satisfies Maxwell's equation. Direct calculation of the vector shape is not simple, and while the vector is not even uniquely determined, it meets all the conditions that must be satisfied in electromagnetism. Of course, its magnitude corresponds to the magnitude determined by the first equation. We could even derive the vector's orientation from the first equation using Lenz's rule.

The equation of motion for a charged particle in an electromagnetic field is

$$\mathbf{F} = m\mathbf{a} = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)\,.$$

In our case, the particle is a positively charged atom with charge $q = e = 1.602 \cdot 10^{-19}\,\mathrm{C}$ and mass $m_{\mathrm{Xe}} = m_\mu A_{\mathrm{Xe}} = 1.67 \cdot 10^{-27}\,\mathrm{kg} \cdot 137.3 = 2.2 \cdot 10^{-25}\,\mathrm{kg}$. We split the vector equation of motion into three components according to each axis

$$\begin{aligned} \ddot{x} &= \frac{eB_0}{m_{\mathrm{Xe}}} \mathrm{e}^{-\Omega t} \left(-\frac{\Omega}{2} y + \dot{y}\right) , \\ \ddot{y} &= \frac{eB_0}{m_{\mathrm{Xe}}} \mathrm{e}^{-\Omega t} \left(\frac{\Omega}{2} x - \dot{x}\right) , \\ \ddot{z} &= 0 , \end{aligned}$$

where we have split the vector product

$$\mathbf{v} \times \mathbf{B} = B_0 \mathrm{e}^{-\Omega t} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = B_0 \mathrm{e}^{-\Omega t} \begin{pmatrix} \dot{y} \\ -\dot{x} \\ 0 \end{pmatrix}$$

We have obtained the equations of motion for an electron in an electromagnetic field. It is a system of three second-order linear differential equations. The motion in the z-axis is rectilinear, and since the electron is supposed to fly out perpendicular to the axis of symmetry, its velocity v_z is zero. Thus the z coordinate of the electron is also zero throughout its motion.

The problem is with the solution of the other two equations, which are interconnected. In the general solution, we could use some tricks of linear algebra, and we would be able to separate the equations (i.e., to have only one coordinate and its time derivative in each equation). Fortunately, we are able to solve this problem without this challenging procedure. Finally, there is no need for numerical simulation, even though it would be a valid approach for such demanding analytical formulas. Therefore, we also include simple Python code in our solution.

However, we will use one trick. Since the equations contain the product of Ω and coordinates, it is unclear what could be neglected to simplify the equations. Therefore, we will use the substitution $\Omega t = T$, where T is a dimensionless time. An important condition is that at $T = 1$ the magnetic intensity field is e-times smaller than at the beginning. The equations will take the form

$$\begin{aligned} \Omega^2 \frac{1}{\Omega^2} \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} &= \frac{eB_0}{m_{\mathrm{Xe}}} \Omega \mathrm{e}^{-\Omega t} \left(-\frac{1}{2} y + \frac{1}{\Omega} \frac{\mathrm{d}y}{\mathrm{d}t}\right) \quad \Rightarrow \quad \frac{\mathrm{d}^2 x}{\mathrm{d}T^2} = \frac{eB_0}{m_{\mathrm{Xe}} \Omega} \mathrm{e}^{-T} \left(-\frac{1}{2} y + \frac{\mathrm{d}y}{\mathrm{d}T}\right) , \\ \Omega^2 \frac{1}{\Omega^2} \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} &= \frac{eB_0}{m_{\mathrm{Xe}}} \Omega \mathrm{e}^{-\Omega t} \left(\frac{1}{2} x - \frac{1}{\Omega} \frac{\mathrm{d}x}{\mathrm{d}t}\right) \quad \Rightarrow \quad \frac{\mathrm{d}^2 y}{\mathrm{d}T^2} = \frac{eB_0}{m_{\mathrm{Xe}} \Omega} \mathrm{e}^{-T} \left(\frac{1}{2} x - \frac{\mathrm{d}x}{\mathrm{d}T}\right) , \end{aligned}$$

Note the dimensionless factor $\alpha = B_0 e / (m_{\mathrm{Xe}} \Omega) \doteq 0.13$, which is quite small. The acceleration in axis x is initially zero because y and $\frac{\mathrm{d}y}{\mathrm{d}T}$ can be chosen to be zero (this corresponds to the fact that you orient the coordinate system such that the electron comes out in the direction of the x-axis). Then the acceleration in the x-axis is proportional to the velocity and the y-position through the factor α. These are proportional to the velocity through the same α factor and the position in x. So, at least for the beginning of the motion, we can estimate that the acceleration in x is suppressed by a factor α^2 relative to the velocity and position in the same axis. Moreover, it is suppressed exponentially with time.

That brings us to the idea of simplifying both equations. It occurred to us that the acceleration on the x-axis is small, and we can put the velocity $\frac{\mathrm{d}x}{\mathrm{d}T} = V_{x0}$ as a constant. That makes the equation for the acceleration in the y-axis much simpler

$$\frac{\mathrm{d}^2 y}{\mathrm{d}T^2} = \frac{eB_0}{m_{\mathrm{Xe}}\Omega} \mathrm{e}^{-T} \left(\frac{V_{x0}T}{2} - V_{x0} \right) .$$

Integrating by T using partial integration gives the velocity as

$$\frac{\mathrm{d}y}{\mathrm{d}T} = -\frac{eB_0}{m_{\mathrm{Xe}}\Omega} \frac{V_{x0}}{2} \mathrm{e}^{-T} \left(T - 1 \right) + C \,,$$

where C is the integration constant, which can be determined from the condition that at time $T = 0$ the velocity is zero.

We get

$$\frac{\mathrm{d}y}{\mathrm{d}T} = -\alpha \frac{V_{x0}}{2} \mathrm{e}^{-T} \left(T - 1 \right) - \alpha \frac{V_{x0}}{2} \,,$$

which indicates that due to the exponential damping of the acceleration, the motion settles down to a uniform linear motion after a while. We can determine its direction from the direction of the velocity vector. That is simply

$$\tan \beta = \frac{V_y(T = \infty)}{V_x(T = \infty)} = \frac{-\alpha \frac{V_{x0}}{2}}{V_{x0}} = \frac{-Be}{2m_{\mathrm{Xe}}\Omega} \doteq -0.067 \,.$$

Since the angle β was zero at the beginning, the atom deviates from the original direction by $\beta = \arctan(-Be/(2m_{\mathrm{Xe}}\Omega)) = -3.8\,^{\circ}$, hence by approximately four degrees in the negative y-axis direction.

Let's check the validity of our approximation. We will integrate the position in the y-axis with respect to T

$$y = \alpha \frac{V_{x0}}{2} \mathrm{e}^{-T} T - \alpha \frac{V_{x0}T}{2} \,,$$

Substituting into the equation for $\frac{\mathrm{d}^2 x}{\mathrm{d}T^2}$ gives

$$\frac{\mathrm{d}^2 x}{\mathrm{d}T^2} = \frac{\alpha^2 V_{x0}}{4} \mathrm{e}^{-T} \left(-3\mathrm{e}^{-T} T + T + 2\mathrm{e}^{-T} - 2 \right) .$$

By integrating from zero to infinity, we get the change in velocity in axis x as

$$\Delta V_x = \frac{\alpha^2 V_{x0}}{4} \left(-\frac{3}{4} + 1 + 1 - 2 \right) = -\alpha^2 V_{x0} \frac{3}{16} \,.$$

The velocity on the x-axis changes by

$$\frac{\Delta V_x}{V_{x0}} = -\frac{3\alpha^2}{16} \doteq -0.34\,\% \,.$$

Thus, it is clear that our assumption of constant velocity is valid and the approximation possible.

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