Problem III.1 ... creative problem-solving

3 points; průměr 2,98; řešilo 123 studentů

Danka attached a garden hose with an inner diameter of 1.5 cm *to a tap in her dorm room and placed the other end on the edge of a window on the eighth floor,* 23 m *above the ground. What is the necessary volumetric flow rate of the water tap so that Danka can spray a stream of water on the people disturbing the night's silence? They are standing below the window at a horizontal distance* 9 m *from the building. Is Danka able to achieve this if water is being sprayed horizontally from the hose and there is no wind?*

Bonus *Where is the farthest these people can stand so Danka can still spray them if the volumetric flow rate of the tap is* $0.4\,\mathrm{l{\cdot}s^{-1}}$*? Danka can now set the end of the hose so that water sprays at an arbitrary angle to the horizontal plane.*

Danka is annoyed by the noise below the windows at night.

When we imagine the situation, we realize that it is actually a horizontal throw of water, that is gushing out of the hose at some velocity v. We can determine the velocity of the volumetric flow rate Q as

$$v = \frac{Q}{S}\,,$$

where $S = \pi d^2/4$ is the cross-sectional area of the hose. Then we can write down the equations for the coordinates of a mass point moving horizontally

$$x = vt\,,$$

$$y = h - \frac{1}{2}gt^2\,.$$

Here we denote x and y as the horizontal and the vertical coordinates. We will place the origin of the coordinate system at a point on the ground directly below the window. Variable t represents the time that elapses from the moment the water comes out of the end of the hose. $g = 9.81\,\mathrm{m{\cdot}s^{-1}}$ is the gravitational acceleration. The water is ejected from the initial height $h = 23\,\mathrm{m}$. The people Danka wants to hit are on the ground, so their y-coordinate is $y = 0\,\mathrm{m}$. They are from the dormitory $l = 9\,\mathrm{m}$ far, which is also their x-coordinate. So we have two equations with two variables, which are the time of flight of the water t and the velocity v

$$l = vt\,,$$

$$0 = h - \frac{1}{2}gt^2\,.$$

We will express the time from the second equation

$$t = \sqrt{\frac{2h}{g}}\,,$$

and then we will plug it into the first of the equations. Therefore, we will get

$$l = v\sqrt{\frac{2h}{g}}\,.$$

We will express the velocity in terms of the volumetric flow and will get

$$l = \frac{Q}{\frac{\pi d^2}{4}}\sqrt{\frac{2h}{g}}\,.$$

From small adjustments, we will express Q from the formula and substitute the values of the known quantities

$$Q = \frac{l\pi d^2}{4}\sqrt{\frac{g}{2h}} \approx 0.73\,\mathrm{l{\cdot}s^{-1}}\,.$$

So Danka will not succeed in splashing on the rude neighbors from the window because it is impossible to get that much volume flow from a tap at home or in the dormitory.

Bonus

If we want to calculate the maximum range from non-zero altitude, the procedure using derivations is very complicated. Therefore, we will use the concept of the so-called *protective parabola*, which is the set of points beyond which, at a given velocity, the stream of water will not get to with any projectile angle. Now, we will show how to derivate it for the constant initial velocity of the stream of water $v = 4Q/(\pi d^2)$. Firstly, we will write down the dependence of both coordinates of the imaginary mass point on the time of flight for the throw with velocity v at an angle φ as

$$x = v\cos\varphi t\,,$$
$$y = h + v\sin\varphi t - \frac{1}{2}gt^2\,.$$

From these equations, we will try to express at what angle φ we must shoot the mass point to go through a fixed point with coordinates $[x, y]$. From the first equation, we will express the time, which we will plug into the second equation

$$t = \frac{x}{v\cos\varphi}\,,$$
$$y = h + v\sin\varphi\frac{x}{v\cos\varphi} - \frac{1}{2}g\frac{x^2}{v^2\cos^2\varphi}\,.$$

Now, we adjust the equation so that only the tangent of the angle φ is present, which we want to express for the fixed x and y

$$y = h + x\tan\varphi - \frac{1}{2}g\frac{x^2}{v^2}\left(1+\tan^2\varphi\right)\,,$$
$$\frac{gx^2}{2v^2}\tan^2\varphi - x\tan\varphi + y + \frac{gx^2}{2v^2} - h = 0\,.$$

We can see that we have a quadratic equation for the tangent of the projectile angle. Therefore, depending on the coordinates of the point x and y, we get either two solutions (we will go through the point either on the way up or on the way down); or no solution (the velocity is too small and we will not go through the point); or one solution, which is the set of points separating these two regions. For the last set of points will the discriminant of the quadratic equation be zero, so we can easily describe it as

$$x^2 - 4\frac{gx^2}{2v^2}\left(y + \frac{gx^2}{2v^2} - h\right) = 0\,,$$
$$\frac{v^2}{2g} - y - \frac{gx^2}{2v^2} + h = 0\,,$$
$$y = \frac{v^2}{2g} - \frac{gx^2}{2v^2} + h\,.$$

For the set of points bounding the space where we can shoot with a given velocity and where we can not, we will get the following expression. This, as you can see, is the equation of the parabola, which we call a protective parabola because we can not hit any point outside of it. To determine the maximum range, calculate the intersection of this parabola with the ground, so the point where $y = 0$:

$$\begin{aligned} 0 &= \frac{v^2}{2g} - \frac{gx_{max}^2}{2v^2} + h\,,\\ \frac{gx_{\mathrm{max}}^2}{2v^2} &= h + \frac{v^2}{2g}\,,\\ x_{\mathrm{max}}^2 &= \frac{2v^2h}{g} + \frac{v^4}{g^2}\,,\\ x_{\mathrm{max}} &= \frac{v}{g}\sqrt{2hg + v^2}\,. \end{aligned}$$

After substituting the initial velocity, we will get the maximum distance at which Danka can hit noisy people as

$$x_{\mathrm{max}} = \frac{4Q}{\pi d^2 g}\sqrt{2hg + \left(\frac{4Q}{\pi d^2}\right)^2} \doteq 4.9\,\mathrm{m}\,.$$

Thus if the noisy people stand more than 5 m away from the dormitory, they are safe from angry Danka.

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