Problem I.5 ... U-tube again

8 points; (chybí statistiky)

We have a U-tube with length l and cross-sectional area S. We pour volume V of water into the tube. The volume V is large enough that the whole U-turn is filled with water but $Sl > V$. When water levels in both arms of the tube are at rest, we seal one of the arms. What is the period of small oscillations of water in the tube? *Karel went crazy again.*

In steady state, the water levels in both parts of the U-tube will be at the same height. We denote the deviation from this equilibrium position by x. Let's define that x is positive when the level in the closed part of the tube falls. We denote the potential energy of the system at the deviation x by $E_\mathrm{p}(x)$.

It can be shown[1] that the period of small oscillations of a system with potential energy depending only on the deflection is

$$T = 2\pi\sqrt{\frac{m}{E''_\mathrm{p}(x_0)}}\,, \tag{1}$$

where x_0 is the equilibrium position of the deflection (in this case $x_0 = 0$), E''_p is the second derivative of E_p with respect to x, and m is the oscillating mass. Calculating the potential energy may not be easy in this case, so we calculate the force instead. Since $F = -E'_\mathrm{p}$, we can then write $E''_\mathrm{p} = -F'$.

There are two forces acting on water – pressure and gravity. Let us denote the pressure in the closed part by p. There is atmospheric pressure p_a in the open part. The pressure force is obviously $F_p = (p - p_\mathrm{a})\,S$. Since the oscillations are very fast, the compression and expansion of the air will be adiabatic

$$pV_\mathrm{u}^\kappa = p_\mathrm{a}V_0^\kappa\,,$$

where κ is the Poisson constant of air, V_0 and V_u are the initial and instantaneous volume in the enclosed part respectively, and p is the air pressure in the closed part at volume V_u. Air volume of the closed part is

$$V_0 = Sa\,,\quad V_\mathrm{u} = S\left(a + x\right)\,,$$

where a is the initial height of the air in the closed tube. Substituting in the relation for the pressure force gives

$$F_p = (p - p_\mathrm{a})\,S = p_\mathrm{a}\left(\left(\frac{V_0}{V_\mathrm{u}}\right)^\kappa - 1\right)S = p_\mathrm{a}\left(\left(\frac{a}{a+x}\right)^\kappa - 1\right)S\,.$$

The gravitational force is easy to determine – if the water drops x in one part, it must rise x in the other. This creates a block of water $2x$ on one side exerting a force

$$F_g = -2S\rho g x\,,$$

where the negative sign means that the force is acting against the displacement. Adding the two forces gives

$$F = F_p + F_g = \left(p_\mathrm{a}\left(\left(\frac{a}{a+x}\right)^\kappa - 1\right) - 2\rho g x\right)S\,.$$

[1] David Morin: *Introduction to Classical Mechanics.*

Now we can calculate the second derivative of the potential energy

$$E_\mathrm{p}'' = -F' = \left(\kappa p_\mathrm{a} a^\kappa \left(a+x\right)^{-\kappa-1} + 2\rho g\right) S\,,$$
$$E_\mathrm{p}''(0) = \left(\frac{\kappa p_\mathrm{a}}{a} + 2\rho g\right) S\,.$$

Now we need to solve the geometry of the tube and determine a. Suppose our tube has a circular cross section (as most tubes do). Let's denote by r its inner radius and R the radius of the bottom bend. This is actually half of a torus. The formula for the volume of the whole torus is

$$V_\mathrm{t} = 2\pi^2 R r^2\,.$$

In our case, the water floods the entire bend (i.e. half of the torus), which is πR of the length of the tube. The rest of the water (i.e. $V - V_\mathrm{t}/2$) will occupy a section of the tube of length l_v, leaving air

$$2a = l - \pi R - l_\mathrm{v} = l - \pi R - \frac{1}{S}\left(V - \frac{1}{2}V_\mathrm{t}\right) = l - \frac{V}{S}\,.$$

As we can see, the torus has the same volume per unit length as the straight parts of the tube, which simplifies the calculation considerably.

Finally, we determine the mass of water as $m = \rho V$ and by substituting it into (1) we get the final result

$$T = \pi\sqrt{\frac{2V}{S}\left(\frac{\kappa p_\mathrm{a}}{\rho}\left(l - \frac{V}{S}\right)^{-1} + g\right)^{-1}}\,.$$

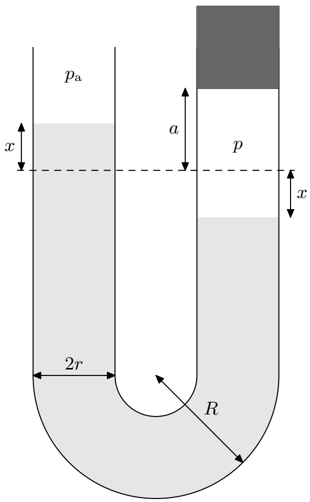

Fig. 1: Parameters of the U-tube.

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