Problem I.3 ... canning jam

6 points; (chybí statistiky)

A cylindrical jar made of glass has a height $h = 7.0\,\mathrm{cm}$ and an inner radius $r = 2.5\,\mathrm{cm}$. We pour hot apricot jam at temperature $T_0 = 80\,\mathrm{^\circ C}$ into the jar, we close the lid and let it cool down. Note that we didn't fill the jar to the top, but left some air between the jam and the lid. If a force of at least $F = 4\,\mathrm{N}$ is applied, a sound is heard as the lid suddenly incurves. We heard this sound $t_\mathrm{i} = 30\,\mathrm{min}$ after the jar had been closed. If jam hardens at temperature $T_\mathrm{h} = 60\,\mathrm{^\circ C}$, was it to be already hard when the lid incurved?

Bonus How long after closing the jar will the jam harden? Assume that the temperature is evenly distributed throughout the jar and that the cooling rate only depends on the difference in temperatures of the jar and its surroundings $T_\mathrm{s} = 25\,\mathrm{^\circ C}$.

Jarda's apricot trees froze this year and he dreams about last year yield.

Firstly, let's discuss what actually happens. When poured into the jar, the jam has a temperature of T_0. The air above it warms to the same temperature – it retains that temperature immediately after the lid is closed. So the jar initially contains an air of temperature T_0 and atmospheric pressure p_0. The jar (and the air inside) then begins to cool down. Before the lid incurves, air volume does not change, and isochoric cooling occurs.

The lid incurves when a force of magnitude F is exerted on it by the difference in air pressure inside and outside the jar

$$F = (p_0 - p_\mathrm{i})\, S \quad \Rightarrow \quad p_\mathrm{i} = p_0 - \frac{F}{S}\,,$$

where $S = \pi r^2$, and p_i is the pressure inside the jar at which the lid incurves. By using Charles's law or the ideal gas law, we obtain

$$T_i = T_0 \frac{p_\mathrm{i}}{p_0} = T_0 \left(1 - \frac{F}{\pi r^2 p_0}\right) \doteq 346\,\mathrm{K} \doteq 73\,\mathrm{^\circ C}\,.$$

Do not forget to use the temperature in kelvins when using the formula. At the moment of the lid incurving, the jam still has a temperature of about $73\,\mathrm{^\circ C}$.

Bonus

The jam will have the temperature calculated above at time t after it begins to cool down. If we know the time evolution of the temperature of the jar, we can find the time at which the jam will reach the temperature at which it hardens. The assumption that the cooling rate is proportional to the difference between its temperature and the temperature of its surroundings can be written in the form of a differential equation

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k\,(T - T_\mathrm{s})\ ,$$

where k is a constant indicating how quickly the jar cools down, and T is the temperature of the jar. Equation can be solved by separation of variables and integrating

$$T = (T_0 - T_\mathrm{s})\, \mathrm{e}^{-kt} + T_\mathrm{s}\,,$$

where the constant before the exponential has to be such that the temperature of the jar and jam at time $t = 0$ is equal to T_0.

The only parameter we do not know in the function $T(t)$ is k. However, we do know the temperature of the glass at time $t_i = 30$ minutes, for which the following holds

$$T_\mathrm{i} = (T_0 - T_\mathrm{s})\,\mathrm{e}^{-kt_\mathrm{i}} + T_\mathrm{s} \quad \Rightarrow \quad k = \frac{1}{t_\mathrm{i}} \ln \frac{T_0 - T_\mathrm{s}}{T_\mathrm{i} - T_\mathrm{s}}\,.$$

If the solidification point is T_h, then

$$T_\mathrm{h} = (T_0 - T_\mathrm{s})\,\mathrm{e}^{-kt_\mathrm{h}} + T_\mathrm{s} \quad \Rightarrow \quad t_\mathrm{t} = t_\mathrm{i} \frac{\ln \frac{T_0 - T_\mathrm{s}}{T_\mathrm{t} - T_\mathrm{s}}}{\ln \frac{T_0 - T_\mathrm{s}}{T_\mathrm{i} - T_\mathrm{s}}}\,.$$

After substitution for the temperature T_i, we get the time at which the jam becomes solid as

$$t_\mathrm{h} = t_\mathrm{i} \frac{\ln \frac{T_0 - T_\mathrm{s}}{T_\mathrm{h} - T_\mathrm{s}}}{\ln \frac{T_0 - T_\mathrm{s}}{T_0\left(1 - \frac{F}{\pi r^2 p_0}\right) - T_\mathrm{s}}} \doteq 98\,\mathrm{min}\,.$$

The apricot jam hardens 98 minutes after closing the lid.

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