Problem I.2 ... weighing an unknown object

3 points; (chybí statistiky)

Let us have an ideal scale which we calibrate using a state standard (etalon) with a mass $m_\mathrm{e} = 1.000\,000\,165\,\mathrm{kg}$ *and a density* $\rho_\mathrm{e} = 21\,535.40\,\mathrm{kg{\cdot}m^{-3}}$*. By calibration, we mean that after placing the standard on the scale, we assign to the measured value the mass* m_e*. The unknown object is weighed under the same conditions in which its volume is* $V_0 = 3.242\,27\,\mathrm{dl}$*. What mass did we measure if we measured the weight* $G = 1.420\,12\,\mathrm{N}$*? What is the actual mass of the object? The experiment is conducted at a place with standard gravitational acceleration* $g = 9.806\,65\,\mathrm{m{\cdot}s^{-2}}$ *and air density* $\rho_\mathrm{v} = 1.292\,23\,\mathrm{kg{\cdot}m^{-3}}$*. Take into account that the calibration is linear, and the unloaded scale shows zero.*

Karel wanted to use a standard.

Before we proceed to the actual solution, let us introduce the following labeling of quantities. There will always be an exact force acting on the scale, which we will designate as F. By G, we will denote the force we measure in the weighing process, by M the mass we measure and m will denote the mass of the object weighed. During the scale calibration, we want to determine the dependence of quantities G and M on the force F. Let us also note that for both the unloaded scale and for the weighed standard, $M(F) = m(F)$ must hold. We will use this as a basis for the calibration.

Our task now will be to find the relations between the above-mentioned quantities. We begin by stating that the following relation between the real force F and the mass m always holds from Archimedes' principle

$$F = F_\mathrm{G} - F_\mathrm{vz}\,,$$
$$F = (m - V\rho_\mathrm{v})g\,, \tag{1}$$

where V is the volume of the weighted object and ρ_v is the density of air. Now we need to find the relations between F and G and then between G and M. Since we only know the relation between M_e and F_e (we do not know the value of G_e), we have to assume something.

Let us consider what calibration means in our case. A measuring instrument that measures some quantity usually works by directly measuring a different quantity and then determining the relation between these two quantities depending on some reference values. It is the determination of this relation that is called calibration.

There are two ways of understanding this task. The first option is that the scale measures the force accurately. Thus, in our established notation, $F = G$ will hold, and during calibration, we want to determine how the weight depends on this force. The scale internally acts as a perfect dynamometer.

The second option is to identify the scale with the dynamometer, which we are yet to calibrate. During the calibration, we want to determine how the measured force G depends on the real force F, and for the weighed mass we will set $G = Mg$.

These two options are very close at first glance, but in reality, there is a significant difference in the calibration process, and in both cases, we will get very different results.

We consider both options to be correct. However, we think that the first solution is a little bit closer to the wording of the problem statement because in the second option, the calibrated instrument is strictly speaking a dynamometer, and in the first option, the calibrated instrument is a scale.

The measured quantity is the mass, and the directly measured quantity is the force (scale calibration)

We have an exactly measured force acting on the scale, and according to the information given in the problem statement, we want to find the function $m(F) = kF + c$. This force involves not only the mass of the weighed object (reduced by a certain factor due to the buoyant force) but also the air column above it. However, since in the unloaded state the force is zero, the sought coefficient c must necessarily be zero, and we no longer need to consider the force of the column of air acting on the scale.

In this case, it is very easy to calculate the real mass of the object – we can go straight from the equation (1) and get

$$m = \frac{G}{g} + V_0\rho_\mathrm{v} \doteq 145.231\,\mathrm{g}\,.$$

What is the difference between this real mass and the mass we get using the calibrated scale? The difference is there because the scale cannot work with the volume of an object.

Let G_e denote the measured weight of the standard. We are looking for coefficient k from the calibration equation. Then we will already be able to find the measured mass of the unknown object using $M = \frac{G}{k}$.

The equation for G_e is set up analogously to the previous case, and we get

$$G_e = V_\mathrm{e}\left(\rho_e - \rho_v\right) g \quad \Rightarrow \quad G_e = m_\mathrm{e} g \left(1 - \frac{\rho_v}{\rho_e}\right)\,.$$

We see that $k = g\left(1 - \frac{\rho_v}{\rho_e}\right)$. Therefore, the calibrated scale will show

$$M = \frac{G}{g\left(1 - \frac{\rho_v}{\rho_e}\right)} \doteq 144.821\,\mathrm{g}\,.$$

The measured quantity is the force from which we calculate the mass (dynamometer calibration)

In this case, we can determine the mass measured directly from the problem statement. While the scale is an (inaccurately) calibrated dynamometer, the mass that we measure for the measured weight G is

$$M = \frac{G}{g} \doteq 144.812\,\mathrm{g}\,.$$

Now we want to find a function $G(F) = k'F + c'$. Using the same arguments as above, we can declare that $c' = 0$ and search further only for k'.

When calibrating with the standard, we set $G_\mathrm{e} = m_\mathrm{e} g$, which is however different from the force F_e that we, again, determine from the equation (1). The coefficient k' is then determined as

$$k' = \frac{G_\mathrm{e}}{F_\mathrm{e}} = \frac{m_\mathrm{e} g}{(m_\mathrm{e} - V_\mathrm{e}\rho_\mathrm{v})g} = \frac{1}{1 - \frac{\rho_\mathrm{v}}{\rho_\mathrm{e}}}\,.$$

Once again, we start from (1) and get

$$m = \frac{F}{g} + V_0\rho_\mathrm{v} = \frac{G}{gk'} + V_0\rho_\mathrm{v} = \frac{G}{g}\left(1 - \frac{\rho_\mathrm{v}}{\rho_\mathrm{e}}\right) + V_0\rho_\mathrm{v} \doteq 145.222\,\mathrm{g}\,.$$

Conclusion and a few remarks

Note that both solutions for the same force F give the same mass M – the result of the problem is different only because the measured force G is different in both options.

Let us also note that scales can only measure accurately for bodies whose density is equal to that of the standard. It should also be noted that all values were given with unnecessarily high precision. This is firstly because of the buoyant force, and secondly, with such a precise measurement, we would have to consider the effect of air currents in the laboratory and many other effects.

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