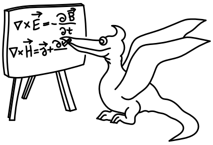

Serial: Waves

Waves are phenomena represented by a number of oscillators distributed in space, exhibiting collective oscillations with predictable dynamics. The physics of waves has a lot in common with the physics of oscillations, and represents the continuation of our ideas about discrete oscillators into continuum systems. For oscillators, we had to derive the dynamical equations of motion. For waves, we will have to derive the so called wave equation. The derivation of this equation will be demonstrated here on a simple example of a string, which will lead us to the definition of some elementary notions necessary to step from discrete systems to continuum.

Taut String

Consider a horizontally taut string. We choose a coordinate system such that the string coincides with the x axis in the equilibrium position and one end of the string is located at the origin of the coordinate system. Therefore, x coordinate corresponds to coordinate along the string. Let the tension in the string be T in the equilibrium position. The tension always acts in the tangent direction. Let the length mass density of the string be λ, so that for a string of length L the overall mass of the string is $m = \lambda L$.

We will assume that the string can only vibrate in the vertical direction. The displacement of the string from the equilibrium position is labeled as $u(x,t)$, since the displacement can differ in both position x and in time t.

We should note that this is very strong assumption. However, we can rationalize it with a simple conception – we assume only very small oscillations, so the tangent direction is always almost parallel with the horizontal axis. That means that the tension in the x direction is almost equal to T along the whole string. Because it acts on each part of the string from both directions with the same magnitude, the parts have no reason to move along the x axis.

Our task is as follows – for a certain displacement profile $u(x,t)$, determine the forces acting on the elements of the string, and hence determine the acceleration of the string elements. We are interested in the force in the direction of the displacement. We will demonstrate how to calculate its change along the x axis.

Let's stop the time for a while and assume $u = u(x)$. An element of the string located in the distance x acts on the surrounding elements with the force T in the tangent direction to the function u. Marking the slope of the tangent from the horizontal direction as φ, $u' = \tan\varphi \approx \varphi$ will apply[1] because, as it was already mentioned, the string is almost horizontal and the angle φ is very small.[2] Here u' stands for the derivative of the function u along a spacial coordinate. The vertical component of this force is $T_y = T\sin\varphi \approx T\varphi$. For an element with the length $\mathrm{d}x$ and the center of gravity in the point x, its right margin will be at point $x + \mathrm{d}x/2$. At this point,

[1]We only need to realize that both of the functions $\tan\varphi$ and u' correspond to the slope of the tangent of the function $u(x)$ at a given point (for a small element $\mathrm{d}x$ and corresponding change $\mathrm{d}u$, $\mathrm{d}u = \tan\varphi\,\mathrm{d}x$ applies).

[2]Really, plot the functions $y = \tan x$ and $y = x$ and you will find out that the difference for small x is negligible. That is the principle of a linear approximation at a given point (replacing the function with a straight line which has the same slope of the tangent as the function).

the tension of $T_y(x + \mathrm{d}x/2)$ acts on it. The similar applies for the left margin. We can calculate the change in the magnitude of the force as the difference between the right and left margin

$$\mathrm{d}F = T_y\left(x + \frac{\mathrm{d}x}{2}\right) - T_y\left(x - \frac{\mathrm{d}x}{2}\right) = T\left(\varphi\left(x + \frac{\mathrm{d}x}{2}\right) - \varphi\left(x - \frac{\mathrm{d}x}{2}\right)\right) .$$

We don't know the function $\varphi(x)$ yet but we can approximate it[3] for a very small a with its tangent

$$\varphi(x + a) \approx \varphi(x) + \varphi'(x)\, a + \dots .$$

We substitute a with $\pm\,\mathrm{d}x/2$, so it will be infinitely small. Substituting we get

$$\mathrm{d}F \approx T\left(\left(\varphi(x) + \varphi'(x)\,\frac{\mathrm{d}x}{2}\right) - \left(\varphi(x) - \varphi'(x)\,\frac{\mathrm{d}x}{2}\right)\right) = T\varphi'(x)\,\mathrm{d}x\,.$$

We have the result[4]

$$\frac{\mathrm{d}F}{\mathrm{d}x} = T\varphi' \approx Tu'' = T\frac{\partial^2 u}{\partial x^2}\,.$$

Let's emphasize again that we worked only with the function $u = u(x)$ with the time stopped. If we are interested in the progress of the system in time, we have to go back to the original function $u = u(x, t)$ and to partial derivatives. We can calculate the acceleration from the Newton's second law. An element with the length $\mathrm{d}x$ has mass $\mathrm{d}m = \lambda\,\mathrm{d}x$, and for the force $\mathrm{d}F = \mathrm{d}m\ddot{u}$ applies, where the dot marks the derivative in respect to time. The resulting wave equation is

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\lambda}\frac{\partial^2 u}{\partial x^2}\,.$$

In this case, the constant $\frac{T}{\lambda}$ will be denoted as v^2. Using dimensional analysis, we can determine that the dimensions of v are that of speed. Constant v indeed corresponds to the so called phase speed of the waves.

The wave equation plays the same role as the equation for the acceleration of the displacement in discrete oscillating systems. We can also find a variant of the natural frequency, but first let's try to find some possible solutions of the wave equation.

Plane Waves

Since waves are built up from individual oscillators, we can try to see whether simple harmonic oscillations can be a solution of the wave equation. Assume that the solution of the wave equation has the following form

$$\hat{u}(x, t) = U(x)\,\mathrm{e}^{-\mathrm{i}\omega t}\,,$$

[3]We are using the linear approximation again. For a general approximation we would use the Taylor series, which you can look up if you are interested. Here we use only the first two terms, as the other are negligible for a small $\mathrm{d}x$.

[4]Working with the function $u(x, t)$ we replace the derivative, e.g. $\frac{\mathrm{d}u}{\mathrm{d}x}$ with the partial derivative, e.g. $\frac{\partial u}{\partial x}$, which is common for multivariable functions. The difference is basically the fact that with the general derivative we would need to consider the dependence of the parameters of the function u (x and t) on each other, while with the partial derivative we do not.

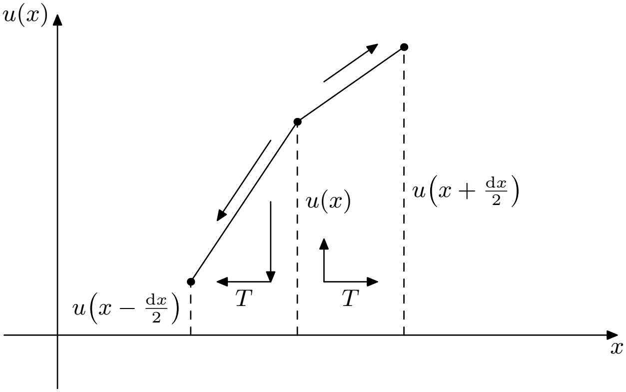

Fig. 1: String is divided into elements of length dx. The forces acting on the element in the middle are shown, including the decomposition into the vertical and horizontal direction.

where $U(x)$ is the profile of the amplitude of the oscillations, which can vary with the position. Again, the displacement is a real variable, but introducing the complex $\hat{u}(x,t)$ leads to a simplification of the algebra. The real solution is recovered as $u(x,t) = \operatorname{Re}\hat{u}(x,t)$. Substitution into the wave equation leads to

$$\begin{aligned} U(x)\,\frac{\mathrm{d}^2\mathrm{e}^{-\mathrm{i}\omega t}}{\mathrm{d}t^2} &= v^2\mathrm{e}^{-\mathrm{i}\omega t}\,\frac{\mathrm{d}^2U}{\mathrm{d}x^2}\,, \\ U(x)\left(-\omega^2\right)\mathrm{e}^{-\mathrm{i}\omega t} &= v^2\mathrm{e}^{-\mathrm{i}\omega t}\,\frac{\mathrm{d}^2U}{\mathrm{d}x^2}\,, \\ \frac{\mathrm{d}^2U}{\mathrm{d}x^2} &= -\frac{\omega^2}{v^2}U(x)\;. \end{aligned}$$

We are familiar with this equation, only the variable in the previous case was time instead of position – this is an equation of simple harmonic oscillations. Therefore, the solution has the form

$$U(x) = A\mathrm{e}^{\mathrm{i}kx}\,,$$

where A is a (potentially complex) constant, and k is a real number. Usually, we refer to k as the *wavenumber*. Substituting this form into the previous equation leads to

$$-k^2A\mathrm{e}^{\mathrm{i}kx} = -\frac{\omega^2}{v^2}U(x) \quad\Rightarrow\quad \omega^2 = k^2v^2\,.$$

This equation is called the *dispersion relation* – it determines the dependence of the frequency of the waves on the wavenumber. Finally, the complex solution of the wave equation is therefore

$$\hat{u}(x,t) = A\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,.$$

The real solution is $u(x,t) = |A|\cos\left(kx-\omega t+\varphi\right)$, where

$$A = |A|\,\mathrm{e}^{\mathrm{i}\varphi}$$

determines both the amplitude $|A|$ and phase shift φ.

Same as was the case for the oscillations, the wave equation is a linear equation, and therefore the solutions of the equation can be constructed by linear superpositions of known solutions. For example, combination

$$\hat{u}'(x,t) = A\mathrm{e}^{\mathrm{i}(kx-\omega t)} + B\mathrm{e}^{\mathrm{i}(-kx-\omega t)}$$

is also a solution of the wave equation.

The behaviour of these so called plane waves can be described as translation of the profile $U(x)$ with passing time t. In order to see this interpretation clearly, consider rewriting the solution as

$$\hat{u}(x,t) = A\mathrm{e}^{\mathrm{i}(kx-\omega t)} = A\mathrm{e}^{\mathrm{i}k\left(x-\frac{\omega}{k}t\right)} = A\mathrm{e}^{\mathrm{i}k(x-vt)}\,,$$

where we used the dispersion relation (and assumed that both ω and k are positive). We can see that the wave therefore moves to the right (in the direction of increasing x). On the other hand, if the solution is in the form

$$\hat{u}(x,t) = A\mathrm{e}^{\mathrm{i}(-kx-\omega t)} = A\mathrm{e}^{-\mathrm{i}k(x+vt)}\,,$$

the waves move to the left (in the direction of decreasing x).

Fourier substitution

In the same way as for oscillations, we can transform the differential wave equation into an algebraic equation. For the solution in the form of the right-moving plane waves, we can write

$$\begin{aligned} \frac{\partial}{\partial t} &\to -\mathrm{i}\omega\,, & \frac{\partial}{\partial x} &\to \mathrm{i}k\,,\\ \frac{\partial^2}{\partial t^2} &\to -\omega^2\,, & \frac{\partial^2}{\partial x^2} &\to -k^2\,. \end{aligned}$$

The application of this substitution leads to a direct derivation of the dispersion relation from the wave equation.

Boundary conditions

Since the waves don't fill out the entire space, the specific solution is constrained by this space in form of the so called boundary conditions. For our example of the string taut between two points, the string does not move at these points. On the other hand, if we had a rope fixed to a pivot on one end and free to move on the other hand, than the restoring force on the free end would be zero. This corresponds to condition

$$\frac{\partial u}{\partial x} = 0$$

at the given point. These points represent the interfaces from which the waves can reflect. For a general interface, we would also observe the transmission of waves through the interface, but in these examples the waves cannot exist beyond this interface, and therefore only the reflection occurs. This can be represented by assuming that the solution is a superposition of two plane waves moving in the opposite direction, possibly with different amplitude and phase shift. The example of such solution follows.

Standing waves

Consider a string taut between two points, where it is kept stationary. The displacement u of the string from the equilibrium position follows the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\lambda}\frac{\partial^2 u}{\partial x^2} .$$

The distance between the points is L. Our task is to determine the stable dynamics of the string, i.e. determine $u(x,t)$, which leads to a repetition of the same cycle. The boundary conditions can be written after definition of the system of coordinates. Let's define this system so that one of the points where the string is kept stationary is the origin of the coordinate system, and the other point lies at the distance L along the x axis. Then, the boundary conditions are

$$u(0,t) = 0 = u(L,t) .$$

Now, assume that the solution can be found in the form of superposition of the two plane waves – one moving to the right and the other moving to the left. Then, the complex displacement is given as

$$\hat{u}(x,t) = A\mathrm{e}^{\mathrm{i}(kx-\omega t)} + B\mathrm{e}^{\mathrm{i}(-kx-\omega t)} .$$

The dispersion relation can be derived from the wave equation

$$\omega^2 = v^2 k^2 ,$$

where $v = \sqrt{\frac{T}{\lambda}}$. The unknowns are therefore A, B and k, since ω is given as $\omega = vk$ (assuming that k is positive – negative k is included in the wave moving in the opposite direction). First boundary condition leads to

$$0 = \hat{u}(0,t) = A\mathrm{e}^{-\mathrm{i}\omega t} + B\mathrm{e}^{-\mathrm{i}\omega t} \quad \Rightarrow \quad A = -B ,$$

the other leads to

$$0 = \hat{u}(L,t) = -B\mathrm{e}^{\mathrm{i}(kL-\omega t)} + B\mathrm{e}^{-\mathrm{i}(kL+\omega t)} ,$$
$$0 = B\left(\mathrm{e}^{-\mathrm{i}kL} - \mathrm{e}^{\mathrm{i}kL}\right) .$$

Using $\mathrm{e}^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x$, we can write

$$0 = B\left(\cos(kL) - \mathrm{i}\sin(kL) - \cos(kL) - \mathrm{i}\sin(kL)\right) ,$$
$$0 = -2\mathrm{i}B\sin(kL) .$$

Hence, we can have either a trivial solution with $B = 0$, or we must have

$$kL = n\pi ,$$

where n is a (positive) integer, which ensures that $\sin(kL) = 0$. The constants that are left undetermined are therefore only the absolute value and phase of B, which corresponds to the amplitude and global phase of the solution. The displacement of the string is therefore given as

$$\hat{u}(x,t) = B\mathrm{e}^{-\mathrm{i}\sqrt{\frac{T}{\lambda}}\frac{n\pi}{L}t}\left(\mathrm{e}^{-\mathrm{i}\frac{n\pi}{L}x} - \mathrm{e}^{\mathrm{i}\frac{n\pi}{L}x}\right) = B\mathrm{e}^{-\mathrm{i}\sqrt{\frac{T}{\lambda}}\frac{n\pi}{L}t}(-2\mathrm{i})\sin\left(\frac{n\pi}{L}x\right) .$$

Using $B = |B|\,\mathrm{e}^{\mathrm{i}\varphi}$ and $-\mathrm{i} = \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}$ leads to

$$\hat{u}(x,t) = 2\,|B|\,\mathrm{e}^{\mathrm{i}\left(\varphi - \frac{\pi}{2}\right)}\mathrm{e}^{-\mathrm{i}\sqrt{\frac{T}{\lambda}}\frac{n\pi}{L}t}\sin\left(\frac{n\pi}{L}x\right)\,.$$

The real displacement is therefore

$$u(x,t) = \mathrm{Re}\,\hat{u}(x,t) = 2\,|B|\cos\left(\sqrt{\frac{T}{\lambda}}\frac{n\pi}{L}t - \varphi + \frac{\pi}{2}\right)\sin\left(\frac{n\pi}{L}x\right) =$$

$$= -2\,|B|\sin\left(\sqrt{\frac{T}{\lambda}}\frac{n\pi}{L}t - \varphi\right)\sin\left(\frac{n\pi}{L}x\right)\,.$$

If we wanted to specify $|B|$ and φ (and perhaps even n), we would need to know the displacement along the whole string at a certain point in time. For example, we could be given that at time $t = 0$

$$u(x,0) = C\sin\left(\frac{\pi}{L}x\right)\,,$$

where C is a known real constant. Therefore, we have

$$C\sin\left(\frac{\pi}{L}x\right) = -2\,|B|\sin(-\varphi)\sin\left(\frac{n\pi}{L}x\right)\,.$$

This means that $n = 1$, $\varphi = \frac{\pi}{2}$ a $|B| = \frac{C}{2}$. The general evolution of the displacement is therefore

$$u(x,t) = -2\frac{C}{2}\sin\left(\sqrt{\frac{T}{\lambda}}\frac{\pi}{L}t - \frac{\pi}{2}\right)\sin\left(\frac{\pi}{L}x\right) = C\cos\left(\sqrt{\frac{T}{\lambda}}\frac{\pi}{L}t\right)\sin\left(\frac{\pi}{L}x\right)\,.$$

This equation does not feature any unknowns, and therefore the dynamics of the string is completely determined. Notice that we required the superposition of two waves in order to describe the dynamics correctly – one wave moving to the right and one to the left. This is the case typical for standing waves, and it represents the reflection at the system boundaries, as mentioned before.

Damping

Damping, i.e. the loss of energy of the waves, can be included in the wave equation through terms including the first order derivatives. These derivatives can be either with respect to time t or position x. Here we present the case where the derivative is with respect to time, but the case for position is similar.

Consider the equation

$$\frac{\partial^2 u}{\partial t^2} + \gamma\frac{\partial u}{\partial t} = v^2\frac{\partial^2 u}{\partial x^2}\,,$$

where γ is the strength of the damping. For the complex displacement, we can carry out the Fourier substitution

$$-\omega^2\hat{u} - \mathrm{i}\gamma\omega\hat{u} = -k^2v^2\hat{u}$$

which means that the dispersion relation is

$$\omega^2 + \mathrm{i}\gamma\omega = k^2v^2\,.$$

This presents us with a non-trivial problem – we need to solve a complex quadratic equation. Mistakes in this derivation can be avoided using the method of completion of the square

$$\omega^2 + \mathrm{i}\gamma\omega - \frac{\gamma^2}{4} + \frac{\gamma^2}{4} = \left(\omega + \mathrm{i}\frac{\gamma}{2}\right)^2 + \frac{\gamma^2}{4} = k^2 v^2 ,$$

$$\left(\omega + \mathrm{i}\frac{\gamma}{2}\right)^2 = k^2 v^2 - \frac{\gamma^2}{4} .$$

Now, we have two possibilities. Either the damping is relatively weak, and we have $k^2 v^2 > \frac{\gamma^2}{4}$. Then

$$\omega + \mathrm{i}\frac{\gamma}{2} = \pm\sqrt{k^2v^2 - \frac{\gamma^2}{4}} \quad \Rightarrow \quad \omega = -\mathrm{i}\frac{\gamma}{2} \pm \sqrt{k^2v^2 - \frac{\gamma^2}{4}} .$$

For strong damping, which is characterized by inequality $k^2 v^2 - \frac{\gamma^2}{4} < 0$, the solution can be written as

$$\omega = -\mathrm{i}\frac{\gamma}{2} \pm \mathrm{i}\sqrt{\frac{\gamma^2}{4} - k^2v^2} .$$

In the first case, the frequency becomes a complex number, and in the second the frequency is completely imaginary for given real k. How should we interpret this value? Let's substitute the value for weak damping into the oscillation part of the plane wave

$$\mathrm{e}^{-\mathrm{i}\omega t} = \mathrm{e}^{-\mathrm{i}\left(-\mathrm{i}\frac{\gamma}{2} \pm \sqrt{k^2v^2 - \frac{\gamma^2}{4}}\right)t} = \mathrm{e}^{-\frac{\gamma}{2}t}\mathrm{e}^{\mp\mathrm{i}\sqrt{k^2v^2 - \frac{\gamma^2}{4}}} .$$

Hence, we can see that the real part of the frequency still corresponds to oscillations, but the imaginary part represents the exponential decay of the amplitude with progressing time. The decay constant in this case is $\frac{\gamma}{2}$, i.e. the stronger the damping, the faster the amplitude of the oscillations decays to zero at the given point.

Similar method could be used for the solution of equation with first derivative in position. This would lead again to the complex quadratic equation, but this time for the wavenumber, which would become complex. There is one more remark left to be made – for a strong damping, the frequency/wavenumber is purely imaginary. This means that the system does not exhibit any oscillations, but only exponential decay in time or position, respectively.

Linearisation

Waves are almost omnipresent in continual physical systems. The reason behind this is very similar to the ubiquity of harmonic oscillations in discrete systems. Close to an equilibrium state, we can often approximate the dynamics of the system as waves.

This process of so called linearisation is carried out as follows. First, we select the variables where we expect the waves to occur. Then, we approximate these variables as small displacements from equilibrium values. For example, for a general variable $u(x,t)$ we could write the approximation as $u(x,t) \approx u_0 + u_1(x,t)$, where u_0 is the equilibrium value and $u_1(x,t)$ is a small displacement from this value in all points and at all times. The specific definition of what it means for the displacement to be small depends on the system in question. For the horizontally taut string, small would mean that the displacement is always much smaller than the length of the string. This approximation is then substituted into our dynamical equation, and we retain only the terms up to the first order in u_1. The resulting equation will be linear and often it will have the form of the wave equation in u_1. The process we just described is rather abstract, and we will try to explain it more in terms of an example inspired by the waves in Bose-Einstein condensate.

Bose-Einstein condensate is a curious state of matter, which can only be attained by ensembles of bosons (specific type of particles) at very low temperatures. We will not explore the specific nature of this state. We will only recognize that we can assign a wavefunction $\psi(x,t)$ to this system, which obeys the so called Gross-Pitaevskii equation

$$-\alpha\frac{\partial^2\psi}{\partial x^2}+\beta\left|\psi\right|^2\psi=\mathrm{i}\hbar\frac{\partial\psi}{\partial t}\,,$$

where α and β are positive real constants, $\hbar$ is a real constant (so called reduced Planck constant) and ψ is generally complex. We will not attempt to derive a truly quantum solution, and we will apply several rather drastic approximations. Assume that there is a stationary solution ψ_0, which is only a function of x and which is real. Then

$$\alpha\frac{\partial^2\psi_0}{\partial x^2}=\beta\left|\psi_0\right|^2\psi_0\,.$$

Now, lets approximate the wavefunction as

$$\psi(x,t)=\psi_0(x)+\psi_1(x,t)\;,$$

where $\psi_1(x,t)\ll\psi_0$. Then

$$-\alpha\frac{\partial^2\psi_0}{\partial x^2}-\alpha\frac{\partial^2\psi_1}{\partial x^2}+\beta\left(\psi_0+\psi_1\right)\left(\psi_0+\psi_1^*\right)\left(\psi_0+\psi_1\right)=\mathrm{i}\hbar\frac{\partial\psi_1}{\partial t}\,,$$

where we used $\left|\psi\right|^2=\psi^*\psi$. To the first order in ψ_1, we have

$$-\alpha\frac{\partial^2\psi_0}{\partial x^2}+\beta\left|\psi_0\right|^2\psi_0-\alpha\frac{\partial^2\psi_1}{\partial x^2}+\beta\left(\psi_0^2\psi_1^*+2\psi_1\psi_0^2\right)=\mathrm{i}\hbar\frac{\partial\psi_1}{\partial t}\,.$$

The first two terms cancel out, following from the equation of the stationary solution. For the remaining terms we carry out the Fourier substitution, which leads to

$$\alpha k^2\psi_1+\beta\psi_0^2\psi_1^*+2\beta\psi_0^2\psi_1=\hbar\omega\psi_1\,.$$

The complex conjugate equation is

$$\alpha k^2\psi_1^*+\beta\psi_0^2\psi_1+2\beta\psi_0^2\psi_1^*=\hbar\omega\psi_1^*\,.$$

The sum of the previous two equations yields the following equation

$$\alpha k^2\left(\psi_1+\psi_1^*\right)+3\beta\psi_0^2\left(\psi_1+\psi_1^*\right)=\hbar\omega\left(\psi_1+\psi_1^*\right)\,.$$

Dividing by $\psi_1+\psi_1^*=2\operatorname{Re}\psi_1$ leads to

$$\alpha k^2+3\beta\psi_0^2=\hbar\omega\,.$$

This equation differs from true dispersion relation for waves in Bose-Einstein condensate, but approaches the correct relation in the limit $\alpha k^2\gg3\beta\psi_0^2$. In this limit, the dispersion relation is quadratic

$$\omega=\frac{\alpha}{\hbar}k^2\,,$$

which is significantly different from the dispersion relation for waves on the string, where we had $\omega=v\left|k\right|$. Similar process can be used to obtain dispersion relations for a large number of systems, where the dynamical equations are known.

What comes next?

Some of these elementary pieces of knowledge about waves will be tested in the current problem series. What will be the topic of our study after that? We will explore the generalisation of normal modes for waves – we will understand the idea of wave polarisation and polarisation vectors. We will also have a look at some more contemporary examples of waves. But, all that only in the next episode of the series.

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