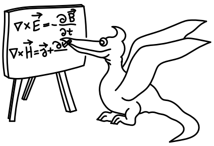


# Serial: Oscillations and Waves

## Introduction

One of most ubiquitous and well described phenomena in nature are oscillations. From guitar strings, pendula or sea waves to oscillations in electric circuits, many systems exhibit some form of this type of motion. What is the origin of oscillations? How do oscillating systems react to external forces? What variables play an important role, and which do not infuence the oscillations? These are the types of questions we will answer in this year's series.

## Back to Springs

If we are to understand more complicated phenomena, such as waves, we will first need to carefully analyse the elementary componenet of any oscillating problem - simple harmonic motion. We will therefore start with the simplest example - a mass on a spring. This system is perhaps even infamous in the world of physics, as we tend encounter it almost always when discussing oscillations. You have probably already met this system, or you are about to meet it during your standard courses on physics. The reason for the common usage of this system in education is the fact, that this innocent looking system already includes most of the important properties shared by all oscillating systems.

There are infact a few properties of oscillating systems which are independent of the specific nature of the system - the properties can be defined for a spring, a pendulum, or any other system. Further on, we will determine what are these properties.

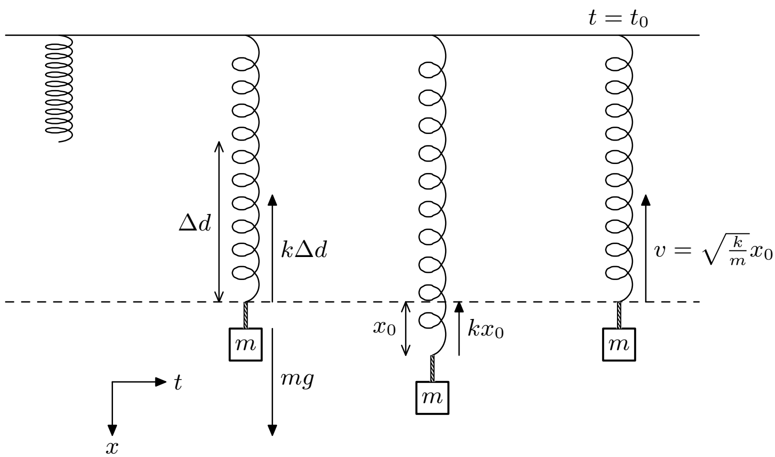


Fig. 1: A mass on a spring before pulling it down, while pulling it down and after release.

Now, consider an ideal spring (see figure 1), which exhibits a a force of magnitude $F = k\Delta d$ when elongated by length $\Delta d$, where $k$ is the so called spring constant. The force tries to restore the original length of the spring. So, when a mass $m$ is hanged on the spring, which is kept stationary, the force of the spring balances the weight of the mass, so it follows that

$$mg = k\Delta d\,.$$

The velocity of the mass is zero, as the mass is at rest, and the kinetic energy of the mass in this state is zero as well. The potential energy of the system is composed from the energy stored in the spring and from the gravitational potential energy. We can choose a reference level of energy as the level when the elongation of the spring is $\Delta d$, and hence even the potential energy in the stationary state can be taken as zero.

What happens when we now displace the mass so that the elongation of the spring is $\Delta d + x_0$? We still assume that the mass is at rest - it is held at certain constant displacement $x_0$. The force on the mass from the spring therefore changes, as well as the potential energy of the system. The change of the potential energy can be expressed as

$$E_\mathrm{p} = \frac{1}{2}kx_0^2\,,$$

as can be checked from the graph of force in the spring as function of the displacement. The force acting on the mass is

$$F = -kx_0\,,$$

, where the minus sign indicates that the force acts in the direction opposite to the direction of displacement $x_0$. Here, we encounter the first two general properties of oscillating systems. Firstly, **the forces in oscillating systems act in the opposite directions to the direction of displacement**. Secondly, **the potential energy of the simple harmonic oscillations increases as square of the displacement**.

Now, we let the oscillations start by releasing the mass in the displaced position. The force accelerates the mass towards desplacement $x(t_0) = 0$, which is reached in time $t_0$. During this motion, the force does work $\frac{1}{2}kx_0^2$ exactly, and so the kinetic energy of the mass at the moment of zero displacement is

$$E_\mathrm{k} = \frac{1}{2}mv^2(t_0) = \frac{1}{2}kx_0^2\,.$$

Clearly, the energy of the system does not disappear, and the mass continues the motion beyond the equilibrium position $x(t_0) = 0$ with speed $v(t_0) = x_0\sqrt{\frac{k}{m}}$, continued by gradual slowing as the force acting on the mass changes the direction. The mass then reaches displacement of $-x_0$, and the process repeats in reverse, until the system reaches the initial state (see figure 1).

## *Towards Generality - Phase Space*

Now, lets try to determine the dynamics of this simple oscillator, that is determine the dependence of different variables on time. For now, we do not use calculus for the derivation of the dynamics, so for those of you who are familiar with derivatives, this approach might seem a bit unnecessarily complicated. but we believe that it is an interesting approach nonetheless. That is because we use the idea of phase space. However, some form of pre-calculus cannot be avoided.

The phase space is a space that consist of two types of variables - variables that correspond to positions/displacements of the system, and variables corresponding to the momenta of the system. In our case the phase space has only two dimensions - one dimension characterising the displacement $x$ and second dimension corresponding to momentum $p = mv$.

During the motion of the oscillator, the law of conservation of energy applies. We say that the energy is a constant of motion (or integral of motion). Hence, we can write

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_0^2\,.$$

This equation can be viewed as a constraint on the values of momentum and displacement in the phase space

$$\frac{p^2}{2m} + \frac{1}{2}kx^2 = \frac{1}{2}kx_0^2\,.$$

Multiplying by $\frac{2}{k}$ leads to

$$\frac{p^2}{km} + x^2 = x_0^2\,.$$

If you recall the lessons on analytic geometry, you can recognize this constraing as a conditions for points lying on the circle of radius $x_0$ in the phase space, where the vertical axis represents variable $\frac{p}{\sqrt{km}}$, see figure 2. So, the oscillator can be described by a point on this circle in the phase space. In order to get the dynamics of the oscillator, we need to determine how the oscillator moves along this trajectory in the phase space. Qualitatively, we can immediately determine that the direction of motion of the oscillator is in the negative, i.e. clockwise direction. This is a consequence of the Newton's second law and the orientation of the force - for positive displacements $x$ the force acts in the negative direction, and so the momentum must increase in the negative direction as well.

For the description of the motion along a circle we know of a useful variable - angular velocity. Generally, this velocity can be time-dependent, and we will denote it as $\omega$. The angular velocity at time $t$ is given by angle $\Delta\varphi$, which the point in the phase space covers in between the time $t$ and $t + \Delta t$, where $\Delta t$ is a small time interval. The definition of angle leads to

$$\Delta\varphi = \frac{\Delta l}{x_0}\,,$$

where $\Delta l$ approaches the arc length covered by the point in time $\Delta t$ for sufficiently small $\Delta t$. This length can be determined by Pythagorean theorem

$$\Delta l = \sqrt{\Delta x^2 + \frac{\Delta p^2}{km}}\,,$$

where the conservation of energy leads to

$$\Delta p = \sqrt{km\left(x_0^2 - (x - \Delta x)^2\right)} - \sqrt{km\left(x_0^2 - x^2\right)}\,,$$

where $\Delta x$ is the change of the variable $x$ in time $\Delta t$je velikost změny souřadnice $x$ za čas $\Delta t$. For small $\Delta t$ a following expansion can be made (see later for details)

$$\sqrt{km\left(x_0^2 - (x - \Delta x)^2\right)} \approx \sqrt{km\left(x_0^2 - x^2 + 2x\Delta x\right)} =$$

$$= \sqrt{km\left(x_0^2 - x^2\right)}\sqrt{1 + \frac{2x\Delta x}{x_0^2 - x^2}} \approx \sqrt{km\left(x_0^2 - x^2\right)}\left(1 + \frac{x\Delta x}{x_0^2 - x^2}\right)\,.$$

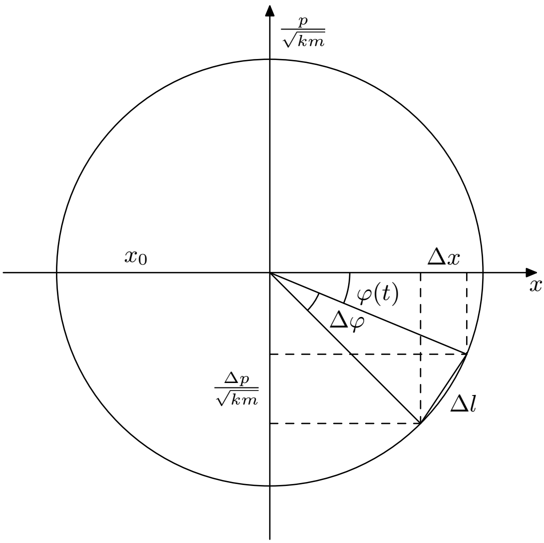


Fig. 2: Trajectory of the harmonic oscillator in the phase space.

Therefore

$$\Delta p = \Delta x \sqrt{km \frac{x^2}{x_0^2 - x^2}}\,,$$

which leads to

$$\Delta l = \Delta x \sqrt{1 + \frac{x^2}{x_0^2 - x^2}} = \Delta x \sqrt{\frac{x_0^2}{x_0^2 - x^2}}\,.$$

Hence, the angle covered by the point is

$$\Delta\varphi = \frac{\Delta l}{x_0} = \Delta x \sqrt{\frac{1}{x_0^2 - x^2}}$$

and the angular velocity is

$$\omega = \frac{\Delta\varphi}{\Delta t} = \frac{\Delta x}{\Delta t} \sqrt{\frac{1}{x_0^2 - x^2}}\,.$$

The ratio $\Delta x/\Delta t$ can be recognized as the (direct) velocity of the mass, which obeys

$$\frac{\Delta x}{\Delta t} = v = \frac{p}{m} = \sqrt{\frac{k}{m}\left(x_0^2 - x^2\right)}$$

and therefore the following critical equation applies

$$\omega = \sqrt{\frac{k}{m}}\,,$$

where we should notice that the angular velocity in the phase space is **time-independent**, i.e. our oscillator moves along the circular trajectory in phase space at constant angular velocity.

This discovery finally leads to an insight into the dynamics of oscillations - the oscillations are given as projection onto the $x$-coordinate axis of the point on the circular trajectory in phase space. This projection is

$$x = x_0 \cos(\varphi) = x_0 \cos(\omega t)$$

in our original setup, where $\varphi$ is the current angle in the phase space, taken in the negative direction (see the figure). Similarly, the momentum can be determined by projection onto the vertical axis as

$$\frac{p}{\sqrt{km}} = -x_0 \sin(\omega t)\,.$$

Therefore,

$$p = -x_0 \sqrt{km} \sin(\omega t)\,,$$

$$v = -x_0 \sqrt{\frac{k}{m}} \sin(\omega t) = -x_0 \omega \sin(\omega t)\,.$$

Finally, from Newton's second law, we can derive that the acceleration of the mass is

$$a = \frac{F}{m} = -\frac{kx}{m} = -x_0 \omega^2 \cos(\omega t)\,.$$

Hence, we recovered the complete dynamics of our oscillator.

What general properties of the system did we uncover in this example? Firstly, the oscillations look as a uniform motion along circular trajectory in the phase space (with usage of proper momentum units). Secondly, the frequency of this motion can be determined from Newton's second law and from the relation between acceleration and displacement

$$a = -\omega^2 x\,.$$

## *Harmonic Approxilator*

We mentioned that an important property of harmonic oscillations is the parabolic profile of the potential energy – potential energy increases as a square of the displacement. It turns out that as long as the system is in a stable position (that in some at least local minimum of potential energy), we can always consider the potential energy to be parabolic for sufficiently small displacements. So, for a stable position, small displacements always lead to emergence of forces acting in the opposite direction to the displacements, and increase linearly with the small displacements.

The linear dependence of the forces on the displacements is necessary for the energetical dependence to be parabolic, and the minus sign signifies the system's tendency to minimise the potential energy and return to the equilibrium minimum position.

In order for the oscillations to be harmonic, it is further required that the kinetic energy, which the system obtains by the action of the emergent forces, increased as a square of the momentum. Overall, we need a circle to be the trajectory of the system in the phase space in order for it to undergo harmonic oscillations. What happens when the energetical dependence is slighlty different will be explored in the next episode of this series.

The most important characteristic of oscillations is their frequency. Generally, the frequency can be determined by analogy to the case of mass on a spring – we start by the determination of the force as a response to small displacement of the system from the equilibrium position. This force is then substituted into Newton's second law, so that the acceleration of the system can be determined. The resulting equation has form $a = -\omega_0^2 x$, where $x$ is the displacement and $\omega_0$ is the frequency we searched for. So, the problem of finding the frequency of oscillations reduces to the problem of approximating the forces acting on the system near the equilibrium position. The simplicity and generality of this method means that many stable systems are modelled as harmonic oscillators for small displacements.

In order to use this method efficiently, it is necessary to remember or derive some approximations of specific functions for small displacements. We start by the elementary functions – polynoms. Consider a small number $x$ (one such that $|x| \ll 1$), for which we are trying to determine the approximate value of expression

$$(1+x)^n \ ,$$

where $n$ is a natural number. Clearly, if $x$ is a small number, then also $\left|x^2\right| \ll |x|$, $\left|x^3\right| \ll \left|x^2\right|$ and so on. Hence, we can talk about the precision of our approximation, based on maximum the order of the $x$ we consider. For example, a second order polynom can be approximated to the first order as

$$(1+x)^2 = 1 + 2x + x^2 \approx 1 + 2x$$

and the polynom of fourth order can be approximated to the second order as

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4 \approx 1 + 4x + 6x^2 \, .$$

For the purposes of the frequency determination for harmonic oscillations, a linear approximation (that is approximation to the first order) will suffice in most cases, i.e. the approximation of type

$$(1+x)^n \approx 1 + cx \, ,$$

where $c$ is some constant. From the binomial theorem for expression $(1+x)^n$, it follows that $c = n$. For the curious ones among you, you can try a proof by induction or you can use the proof of binomial theorem, which makes this statement a simple corollary.

However, this approximation is useful even beyond the set of natural numbers – it can be shown that for a general real number $a$ the same approximation can be made

$$(1+x)^a \approx 1 + ax \, ,$$

to the first order. For example

$$\frac{1}{(1+x)^{\pi}} \approx 1 - \pi x \, .$$

This general statement is harder to prove, if you are curious about it look up the Taylor expansion.

Polynoms are not the only functions we will encounter however. Quite often, we will meeet goniometric functions. How can we approximate these? Lets start by function sine, close to point 0. The value of sine can be interpreted as the ratio of a length of a side to the length of the hypotenuse in a right-angled triangle. Imagine a Thales' circle (see figure 3), where one of the sides of the triangle is very small. The sine of the smallest angle is clearly also a very small number and the length $b$ of the side of the triangle approaches the length of the circular arc $l$, which determines the smallest angle. So, we can say that $b \approx l$ and hence

$$\sin\varphi = \frac{b}{c} \approx \frac{l}{c}\,.$$

We can notice that the central angle subtended by the same arc has value $2\varphi$, and from the definition of angle

$$2\varphi = \frac{l}{\frac{c}{2}} = \frac{2l}{c}\,,$$

which means that

$$\varphi = \frac{l}{c}\,,$$

and therefore

$$\sin\varphi \approx \varphi\,.$$

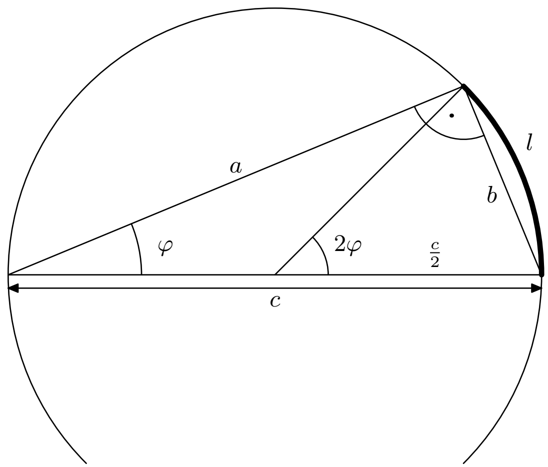


Fig. 3: Thales' theorem for the approximation of value of sine for small angles.

Similar argument leads to

$$\tan\varphi \approx \varphi\,.$$

A perculiar property of cosine is that for a small angle it can be approximated as a constant to the first order in the vicinity of zero (this is related to the fact that cosine has a maximum at zero). To the second order, it can be written that

$$\cos\varphi \approx 1 - \frac{\varphi^2}{2}\,,$$

which is an approximation of the cosine by a parabola.

A final class of functions we will consider are the exponentials and logarithms. We will try to approximate the exponential close to point $x = 0$. As you may know, the Euler's number can be defined as

$$\mathrm{e} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n .$$

Hence we can see, using our polynomial approximation (but we should understand that this is not a rigorous mathematical proof)

$$\mathrm{e}^x = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx} \approx 1+\frac{nx}{n} = 1+x\,.$$

If we need an approximation of a exponential with a different base, we can use relations for logarithms

$$2^x = \left(\mathrm{e}^{\ln 2}\right)^x = \mathrm{e}^{x\ln 2} \approx 1 + x\ln 2$$

and similarly for other bases.

The logarithm itself cannot be expanded around zero, where it approaches $-\infty$. We can however make an approximation around one, which is the inversion of the approximation for the exponential. We are looking for approximation $\ln(1+x)$, and the inversion of exponential approximation leads to

$$\ln(1+x) \approx \ln\left(\mathrm{e}^x\right) = x\,.$$

Finally, for logarithms of different base

$$\log_5(1+x) = \log_5(\mathrm{e})\ln(1+x) \approx x\log_5 \mathrm{e}\,.$$

### *First Approximation - Pendulum*

In order to show the derived approximations in practice, consider a classical pendulum – mass $m$ attached to a massless solid rod of length $l$ under the acceleration of gravity $g$. The rod is attached to an unmoving pivot (see the figure). This system has an equilibrium position – forces on the mass cancel out when the mass is positioned underneath the pivot. For a small displacement from this position, the force of gravity acting on the mass can be projected onto direction along the rod and direction perpendicular to the rod.

From this geometry of the problem, it can be seen that the force perpendicular to the rod has magnitude

$$F_\perp = mg\sin\varphi\,,$$

where $\varphi$ is a small angular displacement of the rod from the equilibrium position. The torque due to this force is

$$M = -lF_\perp = -mgl\sin\varphi$$

and the moment of inertia of the mass with respect to the axis of rotation going through the pivot is

$$J = ml^2\,,$$

and hence the angular acceleration is equal to

$$\alpha = \frac{M}{J} = -\frac{mgl\sin\varphi}{ml^2} = -\frac{g}{l}\sin\varphi\,.$$

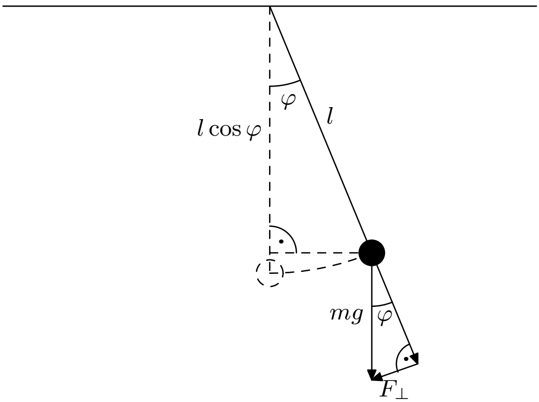


Fig. 4: Geometrie kyvadla při malé výchylce.

For small $\varphi$ we recover the standard equation of harmonic oscillations

$$\alpha \approx -\frac{g}{l}\varphi\,,$$

only instead of linear acceleration and displacement, we have angular acceleration and displacement, but this does not change anything in the algebra of the problem. The system still oscillates harmonically around the equilibrium position with frequency

$$\omega = \sqrt{\frac{g}{l}}\,.$$

Here, we should be careful and recognize that the pendulum itself does not move with constant angular frequency in $\varphi$, i.e. it does not rotate uniformly around the pivot - the angular acceleration changes and generally cannot be set to zero. But, the oscillations in $\varphi$ themselves occur with frequency $\omega$.

We could also check whether the potential energy has the required parabolic profile (kinetic energy for rotating body has a quadratic profile in terms of angular momentum). We can choose the zero level for the potential energy at the height of the equilibrium position of the mass. Then

$$E_\mathrm{p} = mg\left(l - l\cos\varphi\right) \approx mgl\left(1 - \left(1 - \frac{\varphi^2}{2}\right)\right) = \frac{1}{2}mgl\varphi^2\,,$$

and hence the potential energy has the required quadratic profile for small displacements.

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